The Moduli Space of
$N = 2$ Superconformal Field Theories

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ABSTRACT

We review the structure of the moduli space of particular $N = (2,2)$ superconformal field theories. We restrict attention to those of particular use in superstring compactification, namely those with central charge $c = 3d$ for some integer d and whose NS fields have integer $U(1)$ charge. The cases $d = 1$, 2 and 3 are analyzed. It is shown that in the case $d \geq 3$ it is important to use techniques of algebraic geometry rather than rely on metric-based ideas. The phase structure of these moduli spaces is discussed in some detail.

1 Introduction

As well as applications to statistical physics, conformal field theory has proved to be a very powerful tool in string theory. In particular, the ground state of a critical string corresponds to a conformal field theory with a specific central charge. It is of particular interest to classify all such ground states which can therefore be done by finding the space of all conformal field theories of a given central charge. This "moduli space" forms the space of string vacua and may be considered as the stringy analogue of the space of Einstein metrics in general relativity.

The moduli space of conformal field theories thus gives rise to two immediate applications. Firstly one may try to gain an understanding of stringy effects in quantum gravity by comparing the moduli space of conformal field theories with the space of Einstein metrics for a given class of backgrounds. Secondly one may assume that space-time is in the form of flat four-dimensional Minkowski space times some compact part X. The space of possible X's leads to a space of theories of particle physics (i.e., particle masses, couplings, etc.) in four dimensional space time (see, for example, [1]). In this latter case X has a Euclidean signature. Because of the difficulty in analyzing conformal field theories associated to a target space with indefinite signature we will need to restrict our attention to the latter scenario. It should be expected however that many of the features we observe in these lectures should carry over to the former case of stringy quantum gravity of all of space-time.

In section 2 we will deal with simple examples of non-supersymmetric conformal field theories and their moduli space to introduce the basic concepts we will require later in these lectures. The basic example central to a great deal of work in this subject will be that of $c = 1$ theories and the linear sigma model whose target space is a circle. The notion of $R \leftrightarrow \alpha'/R$ duality appears here and will be of some interest later in these lectures.

We will find that extending our ideas to more complicated examples is very difficult to achieve in general. Because of this we are forced to impose restrictions on the type of conformal field theories we study. In particular we want to focus on conformal field theories which are associated to some geometric target space (or perhaps some slightly generalized notion thereof). We also impose that the conformal field theory has N=2 supersymmetry. The effect of this is to force the target space to be a space with a complex structure. In terms of the flat four-dimensional Minkowski space point of view these conditions amount the existence of a space-time supersymmetry. For the purposes of these lectures we may simply regard these conditions as providing us with enough structure to use the tools of algebraic geometry.

In section 3 we will study the superconformal field theory for a sigma model with a complex one-torus as the target space. This will allow us to introduce the complex coordinates which prove to be extremely useful for dealing with later examples.

Section 4 will cover briefly the case of a K3 surface as the target space. In this case we have N=4 supersymmetry. This section will also introduce the concept of a "blow-up" which

is a key construction in algebraic geometry and thus also appears naturally in the context of superconformal field theories. This blow-up also appears to be of central importance to understanding some global issues of the moduli space of N=2 theories and so it will become something of a recurring theme in later sections.

In the sections discussed thus far we will find that using a metric as an effective description of the target space suffices. For the rest of the lectures however we will study examples which require more radical approaches. In particular we will be required to think in terms of algebraic geometry rather than differential geometry.

For the cases we discuss in the later sections, the moduli spaces factor into two parts $\mathscr{M}(X) \cong \mathscr{M}_A(X) \times \mathscr{M}_B(X)$ (moduli some discrete symmetries and so long as we are careful about the boundary points). In geometric terms $\mathscr{M}_A(X)$ corresponds to deformations of the (complexified) Kähler form on X and $\mathscr{M}_B(X)$ corresponds to deformations of the complex structure of X. The factor $\mathscr{M}_B(X)$ turns out to be simple to understand and may be analyzed classically. In order to understand the structure of the moduli space of a particular class of conformal field theories we will have to give three interpretations to each point in $\mathscr{M}_A(X)$:

1. The desired interpretation as a theory with some target space X with a specific Kähler form. This is the most difficult to analyze.

2. A theory with some flat target space containing X with a specific Kähler form. In some limit the fields in this theory are required to live in X. This is the "linear σ-model" of [2].

3. A theory with some space Y, related to X by "mirror symmetry", where the point in moduli space specifies a complex structure on Y.

We will find that the third interpretation in terms of Y provides the simplest context in which to compute the moduli space but that we require the linear σ-model as an intermediary to translate between interpretations on X and Y for each point in this space.

In section 5 we will look at the simplest non-trivial example of the above and explicitly compute $\mathscr{M}_A(X)$. In section 6 we will consider the more general case. Finally in section 7 we present a few concluding remarks.

2   Simple Models without Supersymmetry

We will begin our discussion with the simplest σ-model. For further details and references as well as an excellent introduction to conformal field theory the reader is referred to [3]. Consider a field theory in a two-dimensional space Σ whose action is given by

$$S = \frac{i}{8\pi\alpha'} \int_\Sigma \partial x \bar{\partial} x \, d^2z, \tag{1}$$

where x is real and we are using complex coordinate $z = \sigma_1 + i\sigma_2$ and its conjugate $\bar{z}$ on the world-sheet, Σ.

Classically this action is conformally invariant and under quantization one indeed obtains a conformal field theory with central charge $c = 1$. That is, the stress tensor $T(z)$ which in this case is proportional to the normal ordered product $:\partial x \partial x:(z)$ obeys the operator product expansion

$$T(z)T(w) = \frac{c}{2(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial_w T(w) + \dots \tag{2}$$

with $c = 1$. In this way we say that we have a $c = 1$ conformal field theory description of the real line $\mathbb{R}$.

A simple generalization of (1) is to give the field x an index i which runs $1, \dots, n$. That is, the value of x^i lies in $\mathbb{R}^n$. We may also think of x as defining a map $x: \Sigma \to \mathbb{R}^n$. The stress tensor is now in the form $\sum_i :\partial x^i \partial x^i:(z)$ and we obtain (2) with $c = n$. Thus, at least in this simple case we see that the central charge is a measure of the dimension of the target space of the σ-model.

Returning to the case $n = 1$, we may make an alternative simple generalization of our simple model by imposing periodic boundary conditions on x. That is, we consider the target space to be a circle of radius R which implies $x \cong x + 2\pi R$. Actually it is convenient to rescale x such that $x \cong x + 2\pi$. Then the action is

$$S = \frac{iR^2}{8\pi\alpha'} \int_\Sigma \partial x \bar{\partial} x \, d^2 z. \tag{3}$$

This is a conformal field theory with $c = 1$ for any value of R agreeing with the idea that a circle is a one-dimensional object irrespective of its size! We have a family of field theories parametrized by R.

The key purpose of these lectures is to discuss such families of conformal field theories. The moduli space of theories is the space mapped out by the parameters in the theory. Thus the moduli space for theories on a circle appears at first sight to be the real half line $\mathbb{R}_+$ mapped out by $R > 0$. Certainly this is the moduli space for circles. In order to be sure that we have the right moduli space however it is important to ask the following question. Do two distinct points in the supposed moduli space actually correspond to identical conformal field theories? To answer this question we need to know more about the theory than just the action (3). While different values of R certainly give different actions it may be the case that when we work out the spectrum of fields and their correlators we end with identical field theories with different values of R. Thus turns out frequently to be the case as we shall see.

Consider the local structure of the moduli space around a point corresponding to the action S_0. For a field $\phi(z, \bar{z})$ in this theory we may build a new field theory

$$S = S_0 + g \int_\Sigma \phi \, d^2 z, \tag{4}$$

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for some small g. To maintain conformal invariance we require that the term added be conformally invariant. This implies that ϕ must be of weight (dimension) $(1,1)$ with respect to z and $\bar{z}$. Such an operator is called "marginal". It is important to realize that the operator ϕ belongs to the field theory given by S_0 and not S. In order that we may "transport" this field to a new one in the new theory which is also marginal we require that ϕ should not interact with itself to cause corrections to its own weight [4]. Such a field is called "truly marginal". Such fields naturally span the tangent space to any point in the moduli space.

It turns out that something rather special happens with $R = \sqrt{\alpha'}$ (see [3] for more details). The resultant conformal field theory is completely equivalent to the field theory that corresponds to a string propagating on the group manifold $SU(2)$. This has two striking consequence. Firstly $SU(2)$ is three-dimensional and $c = 1$ thus ruining our initial hope that the central charge might give the dimension of the target space. Secondly, the extra symmetries given by the affine algebra of $SU(2)$ map ϕ to $-\phi$ for this theory. Thus decreasing R appears to be the same as increasing R away from this point in the moduli space.

Actually what we see here is the famous $R \leftrightarrow \alpha'/R$ duality symmetry for a string on a circle [5, 6]. It turns out that the partition function of the string on a circle of radius R is identical to that of a string on a circle of radius α'/R. The reason for this is quite simple to picture. As in normal quantum mechanics, the momentum for the string going around the circle is quantized just like a "particle in a box". This leads to "momentum mode" eigenstates with energy proportional to $m^2\sqrt{\alpha'}/R^2$ for $m \in \mathbb{Z}$. Unlike the quantum theory of a particle we also have "winding modes" where the string wraps around the circle. Clearly the string must wind an integer number, n, times round the circle forcing these modes to be quantized too. The energy of these modes goes like $n^2R^2/(\alpha')^{\frac{3}{2}}$. Thus, the $R \leftrightarrow \alpha'/R$ symmetry appears as an exchange in the rôles of m and n, i.e., an exchange of winding modes with momentum modes. (Note that in order to prove the equivalence of two conformal field theories it is not sufficient just to show that partition functions are identical. In this case however we know that this duality holds near the point $R = \sqrt{\alpha'}$ because of the symmetry $\phi \leftrightarrow -\phi$ and we may integrate along this marginal direction to extend the symmetry to a whole $\mathbb{Z}_2$-symmetry acting on the line of R's. This $\mathbb{Z}_2$ symmetry must be the $R \leftrightarrow \alpha'/R$ duality since this is the only symmetry of the partition function with a fixed point at $R = \sqrt{\alpha'}$.)

At this point our moduli space appears to be the half real line closed at both ends by $R \geq \sqrt{\alpha'}$ with the point at infinity corresponding to $\mathbb{R}$, i.e., a circle of infinite radius. Actually this is not the full moduli space of $c = 1$ conformal field theories. We may form orbifolds of theories we already have. To form an orbifold we divide the theory out by a discrete symmetry G. This not only projects out the non G-invariant states from the original theory but also adds in "twisted" modes corresponding to open strings in the original theory whose ends are identified under G. The circle generically admits a $\mathbb{Z}_2$ symmetry by $x \mapsto -x$. This leads to a new line of theories. We will have more to say about orbifolds later. One may also divide the $SU(2)$ group manifold at the $R = \sqrt{\alpha'}$ point by any discrete subgroup
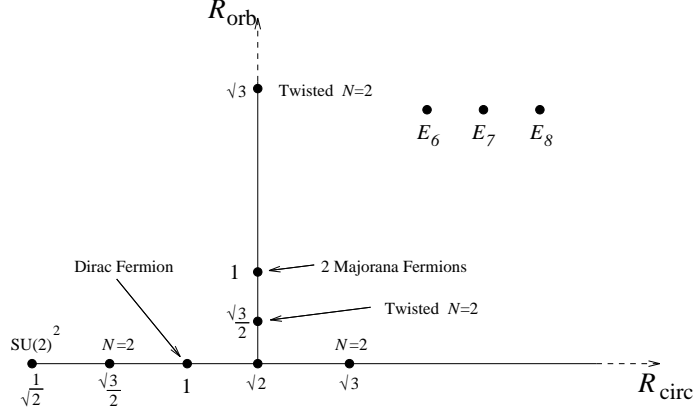
Figure 1: The $c = 1$ Moduli Space (for $\alpha' = \frac{1}{2}$).

of $SU(2)$. Most of these subgroup lead to theories already accounted for but the three exceptional subgroups (the bitetrahedral, bioctahedral and biicosahedral) give new points [7, 8] in the moduli space. Whether the full moduli space of unitary conformal field theories with $c = 1$ has now been accounted for is still to be proven. However partial results towards this end have been reached [9]. The conjectured moduli space for $c = 1$ theories is shown in figure 1. Interesting points have been labeled — the reader is again referred to [3] for more details.

The two generalizations of the real line we have considered thus far were that of $\mathbb{R}^n$ and that of a circle. The next case is to combine these to get a torus $(S^1)^n$. This will give us a conformal field theory of central charge $c = n$. Again, we will not provide any detailed proofs here but only state the results. Narain [10] has shown that a string on an n-torus may be specified by a lattice in a $2n$-dimensional space of indefinite signature (n, n). The generators of the lattice must have inner product in the form of the matrix

$$
\begin{pmatrix}
0 & 1 & & & \\
1 & 0 & & & \\
& & 0 & 1 & \\
& & 1 & 0 & \\
& & & & \ddots
\end{pmatrix}.
\tag{5}
$$

Thus the question of the the moduli space of conformal field theories is reduced to that of finding the moduli space of lattices in the required form. The group $O(n, n)$ acts on any lattice to preserve (5) and this may be taken as the initial guess at the moduli space. We need to identify points in the space $O(n, n)$ corresponding to identical theories which requires us to take a quotient of this group. Firstly we need to divide by $O(n) \times O(n)$. These rotations may be viewed as rotations of the n-dimensional lattice defining the torus itself (or the set

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of allowed winding modes) and the rotations of the dual lattice (giving the set of allowed momentum modes). These rotations have no effect on the underlying field theory. Thus our candidate moduli space becomes the (right) coset $O(n,n)/((O(n) \times O(n))$ (where the group $O(n) \times O(n)$ acts on $O(n,n)$ from the right).

To complete the picture of the moduli space some more identifications are required. These come from elements of $O(n,n)$ which act as automorphisms of the lattice. It is easy to see that such automorphisms must consist of $O(n,n;\mathbb{Z})$, i.e., matrices in $O(n,n)$, preserving (5), which have purely integer entries. Assuming we have accounted for all the necessary identifications our moduli space can then be written as

$$\mathscr{M}_{\text{torus}} \cong O(n,n;\mathbb{Z})\backslash O(n,n)/(O(n) \times O(n)), \tag{6}$$

where the infinite discrete group $O(n,n;\mathbb{Z})$ acts on $O(n,n)$ from the left.

It is an instructive exercise to interpret this space from the point of view of of the world-sheet action [11]. In the case of the circle we could account for the one degree of freedom in the moduli space as corresponding to the radius of the circle. The natural generalization here would be to put a constant metric g_{ij} on the torus. Thus accounts for $\frac{1}{2}n(n+1)$ dimensions but our moduli space (6) has n^2 dimensions. To account for these extra terms one may add the term $i\varepsilon^{\alpha\beta}B_{ij}\partial_\alpha x^i \partial_\beta x^j$ to the action, where α and β indices label coordinates on the world-sheet and B_{ij} is a constant matrix which may be taken to be antisymmetric. Note the factor of i in this term — this is because of the transformation properties of $\varepsilon^{\alpha\beta}$ under the Wick rotation from a Minkowski signature world-sheet to the Euclidean signature we assume in these lectures. Written in terms of complex coordinates on the world-sheet our action then becomes

$$S = \frac{i}{8\pi\alpha'} \int_\Sigma (g_{ij} - B_{ij}) \, \partial x^i \bar{\partial} x^j \, d^2 z. \tag{7}$$

It turns out that in more general cases it is also natural to include the antisymmetric B-term degree of freedom in the field theory.

In order to examine more complicated conformal field theory moduli spaces it will be necessary to impose further structure. The structure we will impose shall be supersymmetry. Thus, from this point on, all conformal field theories considered will be superconformal field theories. The σ-model with target space given by a circle can be extended in a straightforward manner to a superconformal field theory. To do so we add one free left-moving and one free right-moving Majorana-Weyl fermion to our action. Such a free fermion is well-known to correspond to the Ising model and contributes $c = \frac{1}{2}$ to the central charge. Our resulting theory will therefore have $c = \frac{3}{2}$. The moduli space of this theory is at first sight little more complicated than the non-supersymmetric case. The only marginal operator in a generic theory is the one that we already had acting to change the radius of the circle. This should not be surprising since there are no deformations of the free fermion part of the theory. The moduli space for these $c = \frac{3}{2}$ theories contains figure 1 but it actually has more branches. The reader is referred to [12] where this calculation was first performed for

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more details. The extra branches in the moduli space arise because one may put different boundary conditions on the fermions.

This moduli space of superconformal field theories with $c = \frac{3}{2}$ is certainly simpler than the space of all conformal field theories with $c = \frac{3}{2}$ so adding supersymmetry has simplified the problem. Unfortunately this simplification is still not sufficient to study the moduli spaces of non-trivial examples.

The main topic of these lectures will be conformal field theories with $N = 2$ world-sheet supersymmetry. We insist on $N=2$ superconformal invariance in the both sectors and so such theories are often referred to as $N=(2,2)$ superconformal field theories. It is this extended supersymmetry structure which allows for a dramatic simplification of the problem. The fundamental reason for this is that in this case one works exclusively with objects taking values in the complex numbers $\mathbb{C}$ rather than $\mathbb{R}$. Holomorphicity then allows many problems to be solved.

3 The Complex Torus

3.1 The Analytic Approach

To begin our discussion of $N=2$ theories let us try to build the simplest theory with a non-trivial moduli space in analogy with the $c = 1$ theories and the $N = 1, c = \frac{3}{2}$ theories above. Adding a second free moving fermion to each sector forces us to add another free boson to complete the $N = 2$ supermultiplet. It appears that our simplest model will have $c = 3$. Actually the moduli space of such theories turns out to be very messy. Extra branches like those that appeared in the $c = \frac{3}{2}$ theories of [12] proliferate in this case. To bring this situation under control let us impose a further condition on our theories.

Each field in our $N=2$ superconformal field theory has charges $(q, \bar{q})$ under the left-moving and right-moving $U(1)$ currents implicit in the $N=2$ algebra. We will insist that all NS fields in our theory will have $q, \bar{q} \in \mathbb{Z}$. Such a constraint appears naturally in string theory[1] — it is required for "spectral flow" to be a symmetry for fields appearing in the partition function. The reader is referred to [13] for more details.

Let us compute the complete moduli space for such conformal field theories with $c = 3$. The following argument was first presented in [14] and we present here only an outline. For the time being we will assume $\alpha' = \frac{1}{2}$ but this will change later. Using the usual notation $T(z)$, $G^{\pm}(z)$ and $J(z)$ for the fields generating the $N=2$ algebra in the left sector, we have

$$J(z)J(w) = \frac{1}{(z-w)^2} + \dots \tag{8}$$

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[1] Actually the constraint imposed in superstring theory is that these charges be *odd* integer. For the purposes of these lectures however we will not need this stronger condition.

This implies there exists some free boson ϕ such that

$$J(z) = i\partial\phi(z). \tag{9}$$

Furthermore, we can define fields, $\hat{G}^{\pm}$, which are neutral under this $U(1)$ current:

$$G^{\pm} = \hat{G}^{\pm} :e^{i\phi}:. \tag{10}$$

It follows that

$$\hat{G}^+(z)\hat{G}^-(w) = \frac{2}{(z-w)^2} + \dots \tag{11}$$

This in turn implies the existence of a complex free boson H such that

$$\hat{G}^+(z) = \sqrt{2}\partial H, \quad \hat{G}^-(z) = \sqrt{2}\partial H^{\dagger}. \tag{12}$$

The free boson ϕ by itself forms a $c = 1$ conformal field theory as we described earlier. Precisely which conformal field it corresponds to is given by the periodicity of the field ϕ — i.e., the radius of the circle on which ϕ lives. This is fixed by our quantization condition on the fields in our theory. A field of charge q will contain a $:e^{iq\phi}:$ factor. The existence of charge one fields implies that the radius is 1. From figure 1 we see that this corresponds to a Dirac fermion. That is, we have two fermions with matching boundary conditions. Call this a complex fermion ψ. We then have that

$$J(z) = :\psi^{\dagger}\psi:(z). \tag{13}$$

All our degrees of freedom in the moduli space appear to be encapsulated in the field H. The first possibility to consider is that this corresponds to some two-dimensional torus. The moduli space of such $c = 2$ conformal field theories was studied in [15]. Actually we already know from we what have said that it should be

$$O(2,2;\mathbb{Z})\backslash O(2,2)/(O(2) \times O(2)). \tag{14}$$

Neglecting factors of $\mathbb{Z}_2$, one knows that

$$O(2,2)/(O(2) \times O(2)) \cong [Sl(2)/U(1)]^2. \tag{15}$$

We may map $Sl(2)/U(1)$ into the upper half complex plane by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{ai+b}{ci+d}, \quad \text{where} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sl(2). \tag{16}$$

Thus, modulo issues of discrete group actions our moduli space is two copies of the upper-half complex plane. This space is referred to as the "Teichmüller space". The actual moduli

space is the Teichmüller space divided by some discrete group action. The group $O(2,2;\mathbb{Z})$ contains $Sl(2,\mathbb{Z}) \times Sl(2,\mathbb{Z})$. These $Sl(2,\mathbb{Z})$ groups act on the two upper half planes by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \mapsto \frac{az+b}{cz+d}, \quad \text{where} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sl(2,\mathbb{Z}). \tag{17}$$

Thus in our moduli space each upper half plane is divided by this modular group. To picture the moduli space we should therefore consider one fundamental domain for this group for each upper-half plane.

In our haste in building the moduli space thus far we have neglected a few $\mathbb{Z}_2$ factors. The correct inclusion of these divides the space by two further $\mathbb{Z}_2$-actions (see [16] for this calculation). Let us label the two half planes by $\sigma, \tau \in \mathbb{C}; \mathrm{Im}(\sigma), \mathrm{Im}(\tau) > 0$. As well as the modular group action (17) on each of these parameters we also have

$$\begin{aligned} \mu &: (\sigma, \tau) \mapsto (\tau, \sigma), \\ \xi &: (\sigma, \tau) \mapsto (-\bar{\sigma}, -\bar{\tau}). \end{aligned} \tag{18}$$

The resulting moduli space is shown in figure 2.

When the moduli space of $c = 2$ theories was investigated in [15] it was noted that figure 2 was not the complete moduli space. In general one could take orbifolds of the 2-torus to obtain other conformal field theories with $c = 2$. In the context of our $c = 3$ superconformal field theory any action of a group on this torus which is not simply a translation would act non-trivially on $G^{\pm}(z)$. Thus dividing by such a group would ruin the $N{=}2$ superconformal invariance. One concludes therefore that the moduli space as shown in figure 2 includes all valid $N = 2, c = 3$ superconformal field theories.

It will prove extremely useful to analyze figure 2 from the point of view of complex geometry. Let us first note that a 2-torus is a complex manifold of one dimension. Now rewrite the action (7) in terms of complex coordinates x^i on the target space:

$$S = \frac{i}{4\pi\alpha'} \int \left\{ g_{i\bar{j}} (\partial x^i \bar{\partial} x^{\bar{j}} + \bar{\partial} x^i \partial x^{\bar{j}}) - iB_{i\bar{j}} (\partial x^i \bar{\partial} x^{\bar{j}} - \bar{\partial} x^i \partial x^{\bar{j}}) \right\} d^2 z. \tag{19}$$

In terms of the action (19) we appear to have the following degrees of freedom: complex structure on X and a choice of $g_{i\bar{j}}$ and $B_{i\bar{j}}$. Can we interpret the moduli space of figure 2 in terms of these? The complex structure part is easy. It is well-known that the moduli space of a torus is given by the upper half plane divided by $Sl(2,\mathbb{Z})$. We may thus take τ to represent the complex structure in the usual way, i.e., the complex structure is that of a torus constructed from the complex plane by dividing by the translations given by 1 and τ where $\mathrm{Im}(\tau) > 0$.

It may be shown [15] that the complex parameter σ may be built from the remaining degrees of freedom:

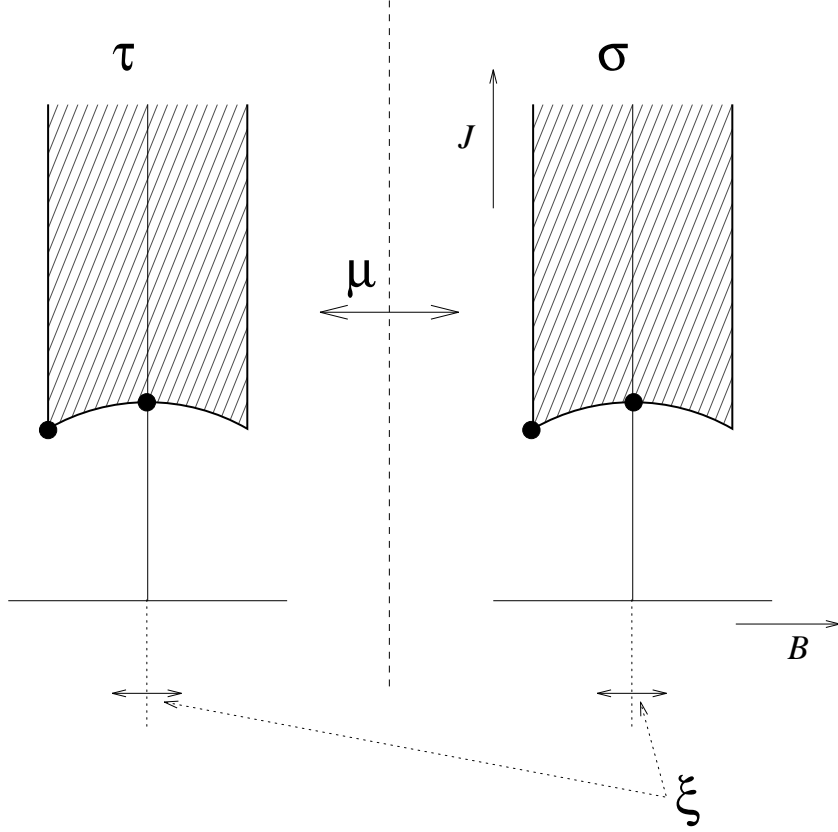$$\sigma = \frac{1}{4\pi^2 \alpha'} (B_1 + iJ_1). \tag{20}$$

Figure 2: The $d = 1$ Moduli Space.

The real numbers B_1 and J_1 describe the degrees of freedom of the metric and the B-term as follows. Introduce a constant 2-form e such that $\int_X e = 1$. Thus e generates $H^2(X, \mathbb{Z})$. Now write

$$B = B_1 e,$$
$$J = J_1 e, \tag{21}$$

where

$$B = \tfrac{i}{2} B_{i\bar{j}} \, dx^i \wedge dx^{\bar{j}},$$
$$J = \tfrac{i}{2} g_{i\bar{j}} \, dx^i \wedge dx^{\bar{j}}. \tag{22}$$

From now on, to avoid cumbersome factors, let us set units so that $4\pi^2 \alpha' = 1$. The element in $Sl(2, \mathbb{Z})$ which takes $\sigma \to \sigma + 1$ is easy to explain as follows. We may rewrite the action (19) as

$$S = S_g + 2\pi i \int_\Sigma x^*(B), \tag{23}$$

where S_g does not depend on B and $x^*(B)$ denotes the pull-back of the two-form, B, from X to the world-sheet Σ via the map $x \colon \Sigma \to X$. First note that this term depending on B is

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"topological" — it depends only on the cohomology class of B. Thus deformations of B do not affect S. This means that the equations of motion do not depend on B. At the quantum level B is important but correlation functions can only "see" B via the expression $\exp(-S)$ in the path integral. Thus, if we shift B by an element of $H^2(X, \mathbb{Z})$ then S shifts by $2\pi i n$ for $n \in \mathbb{Z}$ and thus the correlation functions are invariant. This will be a general symmetry of any theory based on the action (19). In the case of the torus it corresponds to $\sigma \rightarrow \sigma + 1$.

The other generator of $Sl(2, \mathbb{Z})$ not yet accounted for takes $\sigma \rightarrow -1/\sigma$. If we take B to be zero, this amounts to inverting J_1. Since J_1 corresponds to the area of the torus, this action amounts to the $R \leftrightarrow \alpha'/R$ duality acting on the torus. The $\mathbb{Z}_2$-action ξ also has a straight-forward explanation. One can show that it amounts to taking the complex conjugate of the field theory. In target space terms this amounts to taking the conjugate complex structure on the target space and changing the sign of B.

The last symmetry, generated by μ, has no classical explanation. It is "mirror symmetry" (see [13] for an account of the evolution of this phenomenon). For some points in this moduli space we may also view this as coming from $R \leftrightarrow \alpha'/R$ duality. This occurs as follows. Let the torus be generated by dividing the complex plane by translations by $2\pi R_1$ and $2\pi R_2$ in orthogonal directions and let B be zero. This gives

$$
\tau = i\frac{R_1}{R_2}
$$

$$
\sigma = \frac{i}{\alpha'}R_1 R_2.
$$

(24)

The $R \leftrightarrow \alpha'/R$ duality acting on R_2 gives the mirror map μ. It is important to realize however that this is a very special case we will not expect there to be any relationship between mirror symmetry and $R \leftrightarrow \alpha'/R$ dualities in general.

Note that prior knowledge of the existence of mirror symmetry would have been a powerful tool in building this moduli space. Having built the Teichmüller space we could have generated the whole modular group from the following:

1. The modular group for the classical moduli space of complex structures, i.e., $Sl(2, \mathbb{Z})$.

2. The $\mathbb{Z}_2$ generated by the mirror map.

3. The $\mathbb{Z}_2$ generated by complex conjugation plus change in sign of B.

Now let us discuss some of the philosophy behind this latter approach to building the modular group and thus moduli space and contrast it to the method that we used formerly in this section. The first thing to note is that this latter approach differs markedly from the approach one might consider more natural in physics. As physicists we are used to analyzing space-time in terms of a Riemannian metric. Thus when a theory is presented in the form (19) and one is asked the question "What is the moduli space for such theories?", one's

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first reaction will be to find the moduli space of allowed metrics $g_{i\bar{j}}$. In the cases presented thus far in these lectures, the metric is constant and the theory can be solved. Indeed this is the method by which we effectively found the moduli space. As we shall see, when the metric is not so trivial, life becomes considerably more difficult. The stunning feature of the program for building the moduli space using mirror symmetry as outlined in the three steps above is that nowhere did we mention the metric! One might at first think that the complex structure encodes at least part of the metric data since one may think of it as being used to go from the real metric to the Hermitian metric used in (19). Actually this need not be the case as we now explain.

The branch of geometry usually used by physicists is differential geometry. This is implicit when one uses Riemannian geometry. Given a manifold X, one looks very closely at a very small part of it and assumes it looks roughly like flat space. By gluing many such small parts together and defining connections one can build a global picture. There is another method however — namely *algebraic* geometry. In algebraic geometry (over $\mathbb{C}$) one considers recovering information concerning the geometry of a space by considering holomorphic functions on that space. Equivalently one may build a space as the zero locus of some function(s) defined on a simpler space (such as a complex projective space). One need never mention the metric when studying algebraic geometry. Indeed, one need not insist that the space in question is a manifold and thus it may be difficult to define a metric on our space anyway. The important point to notice is that the moduli space of complex structures of a space may be determined using the methods of algebraic geometry.

The situation in $N{=}2$ theories is actually twice as algebraic as classical geometry thanks to mirror symmetry. In most cases (but not always as we see in section 4) the moduli space roughly factorizes into two parts. One part describes the complex structure and the other half gives the Kähler form (i.e., Hermitian metric) and B. Classically the first half can be determined by algebraic geometry whereas the second half is not such a naturally algebraic object. In string theory the mirror map relates the second half to a complex structure calculation allowing both halves to be calculated using algebraic geometry! Certainly within the context of $N{=}2$ theories it would seem unfortunate that physicists have grown to be biased in favour of differential rather than algebraic geometry for analyzing problems. In fact it is the case that not only is algebraic geometry the more convenient setting for analyzing such string theories but that differential geometry can lead to incorrect conclusions as we shall see later. This should not really be a surprise. The fundamental assumption in differential geometry is that if we look closely enough at a small piece of space then it looks like flat space. Why should this be the case in string theory?

3.2   The Algebraic Approach

Let us now reanalyze the moduli space of the torus from an algebraic point of view. This may appear to be unnecessarily cumbersome compared to the analytical approach used earlier in

this section. However, since we make no reference to metrics but rather use mirror symmetry this method will be easier to generalize to more complicated examples.

Consider the complex projective space $\mathbb{P}^2$ with homogeneous coordinates $[x_0, x_1, x_2]$. Define the hypersurface X by the equation

$$f = x_0^3 + x_1^3 + x_2^3 - 3\psi x_0 x_1 x_2 = 0. \tag{25}$$

An object with one complex dimension defined algebraically is called an "algebraic curve". Algebraic geometers tend to name things as if complex dimensions were real dimensions — thus the term "surface" refers to two complex dimensions. The term "algebraic variety" is used for the general case of any dimension. An algebraic curve is a Riemann surface and so the topology is defined by the genus. A curve of genus zero is called "rational" and a curve of genus one is called "elliptic". A straight-forward Euler characteristic calculation (see, for example, [17]) shows that (25) defines an elliptic curve. Thus we have an algebraic description of our torus.

As we vary ψ in (25) we vary the complex structure of X. In general there is a rather subtle relationship between deformations of the polynomial(s) defining the algebraic variety and deformations of complex structure. See [18] for a thorough treatment of this question. In many simple cases however each nontrivial deformation of the defining polynomial (i.e., deformations which cannot be undone by a linear redefinition of the coordinates) define a deformation of complex structure and all deformations of complex structure are obtained this way. This is the case here for the elliptic curve. Note that linear changes of coordinates $(x_0, x_1, x_2) \rightarrow (x_0 + x_1 + x_2, x_0 + \omega x_1 + \omega^2 x_2, x_0 + \omega^2 x_1 + \omega x_2)$ and $(x_0, x_1, x_2) \rightarrow (\omega x_0, x_1, x_2)$, where ω is a on-trivial cube root of unity, induce the following transformations of ψ:

$$\psi \rightarrow -\frac{2 + \psi}{1 - \psi}$$

$$\psi \rightarrow \omega\psi. \tag{26}$$

The moduli space for complex structures of an elliptic curve may be parametrized by ψ modulo the transformations (26).

Note that something special happens at $\psi = 1$. In this case f factorizes:

$$f = (x_0 + x_1 + x_2)(x_0 + \omega x_1 + \omega^2 x_2)(x_0 + \omega^2 x_1 + \omega x_2). \tag{27}$$

Thus $f = 0$ has three components given by the vanishing of each factor. Each factor is a hypersurface defined by a linear equation in $\mathbb{P}^2$ which is a rational curve. Thus X consists of three rational curves. Each pair of rational curves intersect at one point making a total of three points of intersection. This space clearly is not a manifold. In general singularities will occur along a hypersurface defined by f when there is a solution to the equations

$$f = \frac{\partial f}{\partial x_0} = \frac{\partial f}{\partial x_1} = \ldots = 0. \tag{28}$$

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In this case we have solutions when $\psi = 1, \omega, \omega^2, \infty$. These four solutions are mapped to each other by (26). This value of ψ is not really allowed in the moduli space for the elliptic curve since it does not give a smooth elliptic curve. For many purposes however it is useful to add it in to form a compactified moduli space.

Let us now relate ψ to the other parameterization of the complex structure using τ from the previous section. Consider two one-cycles γ_0, γ_1 on X that generate $H_1(X)$. If we cut X along these cycles we obtain a parallelogram that can be put on the complex plane. Let us use ξ to denote the complex number parametrized by this plane. If we rescale by a complex number so that one of the edges lies along the line from 0 to 1, the other edge will lie along 0 to τ. We may also insist that $\operatorname{Im} \tau > 0$ (since if this fails, simply exchange the cycles). Thus, if we define $\Omega = f\, d\xi$ as a holomorphic 1-form on X where f is a constant we have

$$\tau = \frac{\int_{\gamma_1} \Omega}{\int_{\gamma_0} \Omega}. \tag{29}$$

That is, τ is defined by the ratio of two "periods". It turns out that such periods satisfy a differential equation known as the Picard-Fuchs equation. In general it is rather cumbersome to set up the machinery for analyzing these differential equations and solutions so here we present only a quick outline of one of the many methods of derivation of the periods.

First let us find a representative for Ω. In an affine patch where we put $x_0 = 1$ and use x_1 and x_2 as affine coordinates we let

$$\Omega = \left(\frac{\partial f}{\partial x_2} \right)^{-1} dx_1. \tag{30}$$

This is clearly a holomorphic 1-form. One may also show that it is everywhere finite by going to the other patches. Thus it may be used to represent Ω. We now follow [19] in finding the periods. Consider the 1-cycle γ_0 in the elliptic curve defined by $x_0 = 1$, $|x_1| = \epsilon_1$ and x_2 by the unique solution to (25) such that $x_2 \to 0$ as $\psi \to \infty$. Here, ϵ_1 is a small positive real number. This 1-cycle is enclosed by the 2-cycle Γ_0 defined by $x_0 = 1, |x_1| = \epsilon_1, |x_2| = \epsilon_2$ defined in the ambient $\mathbb{P}^2$. We may thus use Cauchy's theorem to give

$$\begin{aligned}
\varpi_0 &= \int_{\gamma_0} \Omega \\
&= \int_{\gamma_0} \left(\frac{\partial f}{\partial x_2} \right)^{-1} dx_1 \\
&= \int_{\Gamma_0} \frac{dx_1\, dx_2}{f} \\
&= \int_{\Gamma_0'} \frac{dx_0\, dx_1\, dx_2}{f}.
\end{aligned} \tag{31}$$

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We have neglected overall constant factors at each stage since these are irrelevant. The last step in (31) is a somewhat formal manipulation to make things more symmetric. Γ_0' is the 3-cycle in $\mathbb{C}^3$ defined by $|x_i| = \epsilon_i$.

Let us now rewrite the defining equation in a more general form which allows us to lift questions into $\mathbb{C}^3$:

$$\varpi_0 = \int_{\Gamma_0'} \frac{dx_0\, dx_1\, dx_2}{a_0 x_0^3 + a_1 x_1^3 + a_2 x_2^3 + a_3 x_0 x_1 x_2}. \tag{32}$$

It is now a simple matter to show that

$$\frac{\partial^3}{\partial a_0 \partial a_1 \partial a_2} \varpi_0 = \frac{\partial^3}{\partial a_3^3} \varpi_0. \tag{33}$$

Actually this last differential equation did not depend on our choice of cycle and will be satisfied by any of the cycles. Thus all periods ϖ may be obtained this way.

Careful analysis [20] of the relationship between this affine point of view and our desired interpretation in the projective space shows that $\varpi = a_3^{-1} f(z)$, where $z = -27 a_0 a_1 a_2 / a_3^3 = \psi^{-3}$ for some function f. For any period on our elliptic curve we thus obtain

$$z \frac{d}{dz} \left\{ \left(z \frac{d}{dz} \right)^2 f - z \left(z \frac{d}{dz} + \tfrac{1}{3} \right) \left(z \frac{d}{dz} + \tfrac{2}{3} \right) \right\} f = 0. \tag{34}$$

This equation will have three independent solutions although we expect only two, since we only have two linearly independent 1-cycles. The extra solution arises because we did the analysis in $\mathbb{C}^3$. The extra solution may be removed by omitting the initial $z\frac{d}{dz}$ in (34). (This may be shown by analyzing the monodromy of the extra solution.) The reader is referred to [21] for an alternative derivation. The resulting second order ODE is the Picard-Fuchs equation and is a hypergeometric differential equation. A general solution is of the form $f = A f_A(z) + B f_B(z)$ where

$$
\begin{aligned}
f_A(z) &= 1 + \tfrac{2}{9}z + \tfrac{10}{81}z^2 + \tfrac{560}{6561}z^3 + \ldots \\
f_B(z) &= f_A(z).\log z + \tfrac{5}{9}z + \tfrac{19}{54}z^2 + \ldots
\end{aligned} \tag{35}
$$

The value $z = 0$ corresponds to our singular elliptic curve and so to map to the fundamental domain as shown in figure 2 we map this to $\tau = i\infty$. This point may be considered the fixed point of the $\tau \to \tau + 1$ symmetry. To recover this symmetry when expressing τ as a ratio of periods one is forced to choose

$$
\begin{aligned}
\tau &= \frac{1}{2\pi i} \left(\frac{f_B(z)}{f_A(z)} + k \right) \\
&= \frac{1}{2\pi i} \left(\log z + k + \tfrac{5}{9}z + \tfrac{37}{162}z^2 + \tfrac{2669}{19683}z^3 + \ldots \right),
\end{aligned} \tag{36}
$$

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for some constant k.

In order to determine k we need to be able to identify another point in our moduli space. For $\psi = 0$ the elliptic curve admits a $\mathbb{Z}_3$ symmetry generated by $(x_0, x_1, x_2) \mapsto (\omega x_0, x_1, x_2)$. This action has three fixed points: $[0, -1, 1], [0, -\omega, 1], [0, -\omega^2, 1]$. The only torus that admits a $\mathbb{Z}_3$ symmetry with fixed points occurs for $\tau = \omega$. In this case the symmetry is generated by $\xi \mapsto \omega\xi$. Therefore we see that $\psi = 0$ is equivalent to $\tau = \omega$. Unfortunately the series (35) fail to converge for $|z| > 1$. In order to extend to this region we need to analytically continue these functions. This is done by representing the functions as Barnes integrals:

$$f_A = \frac{1}{2\pi i} \int_C \frac{\Gamma(3s+1)\Gamma(-s)}{\Gamma^2(s+1)} \left(-\frac{z}{27}\right)^s ds$$

$$f_B = \frac{1}{2\pi i} \int_C \frac{\Gamma(3s+1)\Gamma^2(-s)}{\Gamma(s+1)} \left(\frac{z}{27}\right)^s ds + f_A.\log 27,$$

(37)

where C is the contour running from $-\epsilon - i\infty$ to $-\epsilon + i\infty$ along $\mathrm{Re}(s) = -\epsilon$ for some positive real number ϵ. Closing the contour to the right recovers (35). Enclosing to the left recovers different series valid for $|z| > 1$ which are the analytic continuations of (35). Taking the limit $z \to \infty$ determines $k = -\log 27$.

To complete our moduli space for the $N=2$ theories we copy the above structure. We introduce a new algebraic parameter y and set

$$B_1 + iJ_1 = \sigma = \frac{1}{2\pi i} \left(\log \frac{y}{27} + k + \tfrac{5}{9}y + \tfrac{37}{162}y^2 + \tfrac{2669}{19683}y^3 + \dots\right),$$

(38)

as the mirror of our complex parameter τ. Mirror symmetry now exchanges y and z and hence σ and τ. Simultaneous complex conjugation of y and z sends $\tau \to \bar{\tau}$ and $B \to -B$ as desired.

We have thus built a complete description of the moduli space without requiring a metric on the target space.

4   The K3 Surface

Let us consider a target space that is not flat. That is, we allow $g_{i\bar{j}}$ and $B_{i\bar{j}}$ in (19) to vary over X. It is not possible, in general, to solve this model exactly. This model is known as the non-linear σ-model. One way of analyzing this model is to look at the β-functions in perturbation theory [22]. For conformal invariance these beta functions must vanish. Suppose R is some characteristic radius of the space X is some vague sense. One will expect the perturbation theory to be an expansion roughly in α'/R^2. Thus this method should be reliable in the "large radius" limit.

A simple solution to the vanishing of the β-functions at one loop is given by [22]

$$dB = 0$$
$$R_{i\jmath} = 0, \tag{39}$$

and we demand that the metric $g_{i\bar\jmath}$ be Kähler. That is, it may be written in terms of a closed (1,1)-form J given by (22). A manifold which admits a Ricci-flat Kähler metric is called a "Calabi-Yau manifold".[2] The first condition $dB = 0$ may be thought of as an assumption from which the Calabi-Yau conditions follow. As an alternative one might introduce a non-trivial field $H = dB$, indeed one may allow H to be a cohomologically non-trivial 3-form so that B is only locally defined. WZW models on a group manifold [23] are an example of this. We will not concern ourselves with such models in these lectures mainly because they are very difficult to analyze for anything but the simplest metric. One might also hope that any such model is equivalent to some Calabi-Yau model (if the class of Calabi-Yau spaces is generalized in some way).

Therefore, for the purposes of these lectures, if an $N{=}2$ theory has any large radius interpretation then it must be a Calabi-Yau space. The constraint of Ricci-flatness is actually topological in nature. The first Chern class of the tangent bundle on a manifold, which is an element of $H^2(X, \mathbb{Z})$, is defined in terms of the Ricci curvature and so must be trivial for a Calabi-Yau space. In these lectures we will denote this condition by $K = 0$.[3] Actually, due to a theorem of Yau [24], this argument works in reverse. That is, given a manifold with $K = 0$ one may prove the existence of a Ricci-flat Kähler metric. To be more specific, given a Calabi-Yau manifold (with a given complex structure) and a suitable cohomology class in $H^{1,1}(X)$ there is a *unique* Ricci-flat metric such that the Kähler form is a representative of the cohomology class. Thus rather than concerning ourselves with details of the metric one can specify the cohomology class of J and let Yau's theorem handle the rest.

On the face of it, the non-linear σ-model with target space X, has the following degrees of freedom.

1. The complex structure of X.

2. The cohomology class of the Kähler form, J.

3. The cohomology class of B modulo $H^2(X, \mathbb{Z})$.

Let us use d to denote the complex dimension of the target space. For $d = 1$ the Calabi-Yau space is the torus and we studied that in the last section. For $d = 2$ there are

---

[2]In these lectures we will often allow degenerations of such a manifold. We therefore often use the term "space" rather than "manifold".

[3]This notation comes from algebraic geometry where K denotes the canonical divisor. This divisor is associated to the first Chern class of the canonical bundle which is turn is equal to the first Chern class of the cotangent bundle.

two possibilities. Firstly there is the two complex dimensional torus — again a flat space. Secondly one might have a "K3 surface". Before moving on to $d = 3$ we will briefly analyze some aspects of this K3 case. Although this is not entirely relevant for the case $d = 3$ it will allow us to introduce Calabi-Yau orbifolds which will be of some importance later on.

In the case $d = 1$ we found that the superconformal algebra provided a very restricting set of conditions for the theory. It was this that forced the target space to be a torus. For $d \geq 2$ we lose some of the power of this method but for $d = 2$ one may still impose certain restrictions. In this case we can define the boson $\phi(z)$ by

$$J(z) = i\sqrt{2}\partial\phi(z). \tag{40}$$

We also have the fields

$$\Omega^{\pm}(z) = :e^{\pm i\sqrt{2}\phi}:(z), \tag{41}$$

which are present in any non-trivial theory with NS fields with integer $U(1)$ charges. The fields $J(z), \Omega^{\pm}(z)$ together form an $SU(2)$ affine algebra [25]. Thus the $U(1)$ of the $N{=}2$ superconformal algebra has been elevated to $SU(2)$. In fact, the two fields $G^{\pm}(z)$ may be split up into 4 fields transforming as a $\mathbf{2}+\mathbf{2}$ representation of $SU(2)$. In this way, the algebra is extended to an $N{=}4$ superconformal algebra. That is to say, in the case $d = 2$, any $N{=}2$ superconformal field theory with NS fields with integer $U(1)$ charges is automatically an $N{=}4$ superconformal field theory.

This $N{=}4$ supersymmetry has a striking effect on the perturbation theory for the non-linear σ-model. One may show [26, 27, 28] that there are no corrections to (39) at any loop order, nor indeed any nonperturbative corrections. One may also show that the target space has a quaternionic structure as well as a complex structure.

Now let us turn our attention to K3 itself. What exactly is a K3 surface? One way of building one is in the form of an orbifold. Consider the two complex dimensional torus constructed by taking a quotient of $\mathbb{C}^2$, with coordinates (ξ_1, ξ_2), by the translations $(1, 0), (i, 0), (0, 1), (0, i)$. In other words we take the product of two tori with $\tau = i$ as defined in the last section. The group $\mathbb{Z}_2$ generated by $g\colon (\xi_1, \xi_2) \mapsto (-\xi_1, -\xi_2)$ fixes the 16 points $(0, 0), (\frac{1}{2}, 0), (\frac{1}{2}i, 0), (\frac{1}{2} + \frac{1}{2}i, 0), (0, \frac{1}{2}), \ldots, (\frac{1}{2} + \frac{1}{2}i, \frac{1}{2} + \frac{1}{2}i)$. This means that when we build the space that is the quotient of the complex two-torus by this $\mathbb{Z}_2$ we will have a space with 16 singularities. Each of the 16 singularities looks locally like $\mathbb{C}^2/\mathbb{Z}_2$ where the $\mathbb{Z}_2$ in the denominator is generated by $(\xi_1, \xi_2) \mapsto (-\xi_1, -\xi_2)$. This latter space has an isolated singularity at the origin.

A space with quotient singularities is known as an orbifold. Because such spaces are not manifolds, naïve application of differential geometry would be inappropriate. They are simple to deal with in terms of algebraic geometry however. To see how such objects appear in our moduli spaces we need to introduce the concept of a "blow-up".

To begin with let us define the space

$$\mathcal{O}(-1) = \{[a, b], (x, y) \in \mathbb{P}^1 \times \mathbb{C}^2; \, ay = bx\}. \tag{42}$$

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Clearly this is a two-dimensional space (there is one constraint in a three dimensional space). If $(x, y) \neq (0, 0)$ then a/b is determined and we fix a point on $\mathbb{P}^1$. Thus, away from the origin of $\mathbb{C}^2$ there is a one-to-one map between $\mathbb{C}^2$ and $\mathcal{O}(-1)$. At the origin of $\mathbb{C}^2$ there is no constraint on $[a, b]$ so we recover the whole $\mathbb{P}^1$. Thus $\mathcal{O}(-1)$ looks like $\mathbb{C}^2$ with the origin removed and replaced by $\mathbb{P}^1$. One may also check that $\mathcal{O}(-1)$ is smooth from (28). The space $\mathcal{O}(-1)$ is said to be obtained from $\mathbb{C}^2$ by "blowing-up" the origin. Given a space, one may produce an infinite set of spaces by blowing-up smooth points. This might at first appear an alarming prospect from the point of view of trying to classify Calabi-Yau spaces but it turns out that if we take a Calabi-Yau space and blow-up a smooth point then the resulting space has $-K > 0$. That is, the resulting space will not be Calabi-Yau (although it is still Kähler). The usefulness of the blowing-up construction in the context of Calabi-Yau spaces stems from what happens when we try to blow-up singular points as we now see.

Let us return the the space $\mathbb{C}^2/\mathbb{Z}_2$. Let us define

$$x = \xi_1^2,$$
$$y = \xi_2^2, \tag{43}$$
$$z = \xi_1 \xi_2.$$

Clearly x, y, z are invariant under the $\mathbb{Z}_2$ action. We also have $xy = z^2$. In fact this defines a one-to-one map between the space $\mathbb{C}^2/\mathbb{Z}_2$ and the hypersurface $xy = z^2$ in $\mathbb{C}^3$. The latter forms the description of the quotient singularity in algebraic geometry. Let us pretend that $[x, y, z]$ are the homogeneous coordinates of $\mathbb{P}^2$ rather than the affine coordinates of $\mathbb{C}^3$. In this case $xy = z^2$ defines a smooth $\mathbb{P}^1 \subset \mathbb{P}^2$. Putting coordinates $[a, b]$ on this $\mathbb{P}^1$ we may map it into $\mathbb{P}^2$ via $x = a^2, y = b^2, z = ab$. The definition of $\mathcal{O}(-1)$ amounted to taking the subspace of $\mathbb{P}^1 \times \mathbb{C}^2$ where the affine coordinates on $\mathbb{C}^2$ represented the homogeneous coordinates on $\mathbb{P}^1$. Let us now play the same game with our subspace of $\mathbb{C}^3$. That is we consider

$$\{[a, b], (x, y, z) \in \mathbb{P}^1 \times \mathbb{C}^3; \ xy = z^2, a^2 z = abx, a^2 y = b^2 x, aby = b^2 z\}. \tag{44}$$

Now, with reasoning similar to before, away from $(0, 0, 0) \in \mathbb{C}^3$ the space looks like $\mathbb{C}^2/\mathbb{Z}_2$ whereas at $(0, 0, 0) \in \mathbb{C}^3$ we have a $\mathbb{P}^1$. That is, we have replaced the singular point of $\mathbb{C}^2/\mathbb{Z}_2$ by a $\mathbb{P}^1$ and in the process we have ended up with a smooth space! Actually the above form of this space can be simplified. The constraints are sufficient to uniquely determine z from the other variables. Thus the space is isomorphic to

$$\mathcal{O}(-2) = \{[a, b], (x, y) \in \mathbb{P}^1 \times \mathbb{C}^2; a^2 y = b^2 x\}. \tag{45}$$

In general the notation $\mathcal{O}(n)$ represents the line bundle over $\mathbb{P}^1$ whose first Chern class, when integrated over the base, equals n. The condition $K = 0$ is only met by $\mathcal{O}(n)$ when $n = -2$.

Thus, it is only when blowing up with $\mathcal{O}(-2)$ that we can hope to obtain a Calabi-Yau space in two dimensions.

Note that blowing-up changes the topology. In particular, introducing the new $\mathbb{P}^1$ adds an element to $H_2(X)$. This new element is called an "exceptional divisor". The term "divisor" means, for our purposes, any linear combination of complex codimension one objects in our algebraic variety. "Exceptional" refers to the fact that it came from a blow-up. We may take our space obtained by dividing the complex 2-torus by $\mathbb{Z}_2$ and obtain a smooth space by blowing up all 16 fixed points. The Hodge numbers $h^{p,q}$ can be calculated by starting with the cohomology of the complex 2-torus, projecting out those elements which are not invariant under the $\mathbb{Z}_2$ action and then adding in the elements from the exceptional divisors of the blow-up. This is captured by the following Hodge diamonds:

$$
\begin{array}{ccc}
\begin{array}{ccccc}
 & & 1 & & \\
 & 2 & & 2 & \\
1 & & 4 & & 1 \\
 & 2 & & 2 & \\
 & & 1 & &
\end{array}
\xrightarrow{/\mathbb{Z}_2}
\begin{array}{ccccc}
 & & 1 & & \\
 & 0 & & 0 & \\
1 & & 4 & & 1 \\
 & 0 & & 0 & \\
 & & 1 & &
\end{array}
\xrightarrow{\mathrm{Blow-up}}
\begin{array}{ccccc}
 & & 1 & & \\
 & 0 & & 0 & \\
1 & & 20 & & 1 \\
 & 0 & & 0 & \\
 & & 1 & &
\end{array}
\end{array}
\tag{46}
$$

The diamond on the left is that of the complex 2-torus and on the right that of the desired space. The smooth Calabi-Yau surface is an example of a K3 surface. Any smooth complex surface which is Calabi-Yau and is not a torus will be diffeomorphic to a K3 surface (see for example [29]). One may divide complex 2-tori by many discrete groups. So long as this discrete group is a subgroup of $SU(2)$ one may blow-up the resulting space and, if there were any fixed points, one will find the same Hodge diamond as above. Blow-ups of complicated quotient singularities are performed by a sequence of adding $\mathcal{O}(-2)$ spaces [29].

Let us now consider the Kähler form on X, when X is a K3 surface. Then since $h^{1,1} = 20$, the cohomology class of the Kähler form lives in the vector space $\mathbb{R}^{20}$. There is a natural linear mapping between $(1,1)$-forms and divisors. This mapping is "the dual of the dual" in the following sense. We take a $(1,1)$-form e_i and find the dual form $\tilde{e}_i$ such that $\int_X \tilde{e}_i \wedge e_j = \delta_{ij}$. C_i is then taken to be the dual of $\tilde{e}_i$ in the sense that $\int_{C_i} \tilde{e}_j = \delta_{ij}$. Let us choose a basis e_i, $i = 1, \ldots, 20$. One choice could be to take $e_{17}, \ldots, e_{20}$ forming a basis of the $(1,1)$-forms of the original two-torus and have $e_1, \ldots, e_{16}$ associated to the 16 exceptional divisors. We then write

$$
J = \sum_{i=1}^{20} = J_i e_i.
\tag{47}
$$

Using standard Kähler geometry (see, for example, [30]) one may calculate the area of a curve, C, given its homology class, i.e., Area $= \int_C J = \int_X e \wedge J$. To calculate this we need to know the intersection form $\langle e_i, e_j \rangle = \int_X e_i \wedge e_j$. Our association between e_i and C_i allows us to rephrase questions in terms of intersection numbers since $\langle e_i, e_j \rangle = \#(C_i \cap C_j)_X$.

Let us consider the intersection numbers of an exceptional divisor, say, C_1, with the other generators. Each exceptional divisor comes from a different fixed point and so need not touch
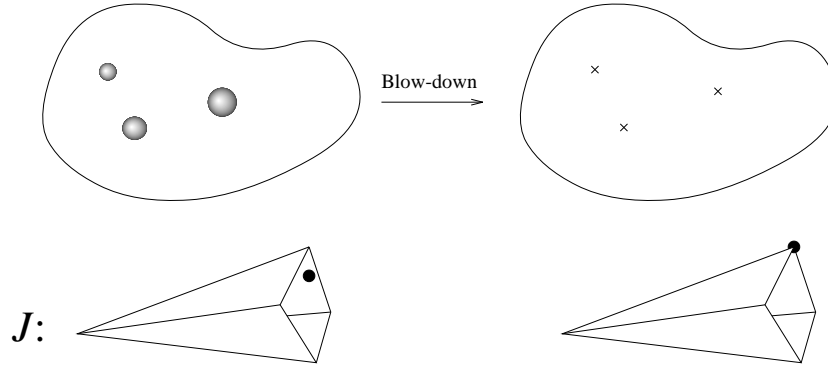
Figure 3: Blowing down K3 and the associated Kähler form.

the others. Also a generator of H_2 on the original complex two-torus may defined such that it does not pass through the fixed points. Thus $\#(C_i \cap C_1)_X = 0$ for $i = 2, \ldots, 20$. What about $\#(C_1 \cap C_1)_X$? Consider the rational curve, C, in $\mathcal{O}(n)$ defined as the base space while considering $\mathcal{O}(n)$ as a line bundle. This bundle may be considered as the normal bundle of the embedding of this rational curve in the space $\mathcal{O}(n)$. Deformations of the curve may then be given by holomorphic sections of this normal bundle. If $n \geq 0$ there are non-zero sections of the bundle with generically n zeroes. In this manner, we see that C has self-intersection n. In our case $n = -2$ and there are no deformations of the rational curve. However one may assume that the above reasoning extends for $n < 0$ and we declare that $\#(C_1 \cap C_1)_X = -2$. We therefore obtain

$$\text{Area}(C_1) = \int_{C_1} J$$
$$= -2J_1. \tag{48}$$

Since it is a reasonable assertion that the area of this curve should be positive, we should therefore impose that $-J_1 > 0$. When one computes all the areas and volumes of all algebraic subspaces of X as well as the volume of X itself, this positivity condition marks out a region of $\mathbb{R}^{20}$ where the Kähler is allowed to have values. The shape marked out is a cone — if J is a valid Kähler form then so is λJ for λ a positive real number. This subspace of $\mathbb{R}^{20}$ is called the "Kähler cone".

Note that when we take the limit $J_1 \to 0$, the area of the exceptional divisor shrinks down to zero. This is precisely the reverse of blowing-up. By sending $J_i \to 0$ for $i = 1, \ldots, 16$ we can thus recover the orbifold of the complex 2-torus. In this sense, the orbifold "lives" in the boundary of the Kähler cone. This is shown in figure 3.

It is worth mentioning that one may analyze the orbifold directly in string theory without having to resolve the singularities [31]. To study the orbifold in A/G, where A is a smooth space and G is a discrete group acting with fixed points, one considers string theory on A. As well as closed strings on A one should also consider strings which are open but whose

ends are identified under the action of G since such strings will be closed on the orbifold. These are called "twisted" strings. One then projects on to all G-invariant states. This procedure allows one to deduce the string spectrum, and hence Euler characteristic, of the orbifold [31]. Applying such a method for our two-torus divided by $\mathbb{Z}_2$ we obtain an Euler characteristic of 24 in agreement with the cohomology of K3. This calculation is actually more surprising than first meets the eye. The orbifold itself, despite being singular may still be triangulated and thus singular homology may be defined on it. The orbifold is a simplicial complex obtained from the two-torus by removing 16 points, dividing by a freely acting $\mathbb{Z}_2$ and adding back the 16 points. The Euler characteristic is thus 8. Thus string theory somehow knew that when it was on the orbifold that the associated geometry was that of a (blown-down) K3 surface rather than the obvious simplicial complex for purposes of cohomology. This is another reason to believe that algebraic geometry is the best setting for understanding the geometry of string theory — blowing up the orbifold to form K3 is a very natural process in algebraic geometry. While the correct geometrical picture is now well-understood for simple cases (see, for example [32]) addition of so-called "discrete torsion" into the orbifolding process can introduce many curiosities [33].

Now let us discuss the deformations of complex structure. Deformations of a complex manifold are given by elements of $H^1(X, T)$, i.e., one-forms with values in the holomorphic tangent bundle. If X is a Calabi-Yau manifold of d dimensions, there is a unique $(d, 0)$-form (up to rescaling) which may be used to form an isomorphism between $H^1(X, T)$ and $H^{d-1,1}(X)$. See [1] for a fuller explanation. It is known [34] that all elements of this cohomology group give deformations assuming X is smooth. Thus the dimension of the moduli space of complex structures is given by the Hodge number $h^{d-1,1}$ which in this case is 20. It is actually rather tricky to explicitly show all 20 deformations in one model. Let us build K3 as an algebraic variety as we did for the elliptic curve. We know that any Calabi-Yau surface we can build that is simply connected will be a K3 surface. One can show that the hypersurface in $\mathbb{P}^3$ defined by an equation of total degree 4 in the homogeneous coordinates $[x_0, x_1, x_2, x_3]$ has the required properties (see chapter 2 of [17] for a thorough account of such constructions). It is not too difficult to show that writing down the most general quartic defining equation and accounting for linear redefinitions of the coordinates leads to 19 deformations of the defining equation. Thus we appear to "miss" one of the the deformations when we analyze the K3 surface in this way.

The reason for this is not too difficult to see. It comes from the fact that the deformations of complex structure and those of the Kähler form tend to "interfere" with each. Suppose we have an algebraic curve C embedded in our K3 surface. To such a curve we associate a two-form $e(C)$ by the linear map discussed earlier. This two-form will be a (1,1)-form by standard results in complex geometry [30]. Since C is an element of $H_2(X, \mathbb{Z})$, $e(C)$ will also be an element of $H^2(X, \mathbb{Z})$. As we vary the complex structure of X, the way that the lattice $H^2(X, \mathbb{Z})$ sits within the Hodge decomposition $H^2(X, \mathbb{C}) \cong H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$ will vary. Thus, the very existence of such a curve C will put constraints on the complex structure.

Our K3 surface does indeed have such a curve — if we define $\mathbb{P}^2 \subset \mathbb{P}^3$ by the vanishing of a generic equation linear in the homogeneous coordinates, then the intersection of this "hyperplane" $\mathbb{P}^2$ with the quartic hypersurface will be an algebraic curve (of genus 3). The fact that there is only class of such curves puts one constraint on the complex structure and lowers the dimension of the moduli space from 20 to 19. In fact the general rule is that if we have n algebraic curves in the variety generating n dimensions of the Kähler cone, we only have $20 - n$ deformations of complex structure. Thus it is hopeless to try to capture the whole moduli space this way.

The actual analysis of the moduli space of K3 surfaces using mirror symmetry is very interesting but we will not pursue it much further here since it is of little relevance to the rest of these lectures. The reader is referred to [35, 36] for further details. What we will present here is the final answer. It turns out the the moduli space fails completely to factorize into complex structure times Kähler parts. The form of the space turns out to be remarkably similar to that of the torus:

$$\mathscr{M}_{\mathrm{K3}} \cong O(4, 20; \mathbb{Z}) \backslash O(4, 20)/(O(4) \times O(20)), \tag{49}$$

where the indefinite metric being preserved is of the form $(-E_8) \oplus (-E_8) \oplus H \oplus H \oplus H \oplus H$, where E_8 is the definite even self dual lattice in 8 dimensions and H is the 2-dimensional matrix $H_{ij} = 0$ for $i = j$ and $H_{ij} = 1$ for $i \neq j$. From Narain's work [10] this is precisely the moduli space for string on a left-right unsymmetric torus of dimension (4,20). Why this should be the case is as yet only partially understood [37].

Note that the moduli space is again of the form of some smooth Teichmüller space divided out by some modular group. Note also that this moduli space includes orbifolds such as our complex two-torus divided by $\mathbb{Z}_2$. It should be the case (but this has not yet been checked) that the modular group should identify at least some of the orbifolds with classically smooth models as predicted in [25].

5   Calabi-Yau Threefolds

5.1   Generalities

Now let us progress on to the case $d = 3$. Recall that for $d = 1$ there was only one Calabi-Yau manifold — the torus. For $d = 2$ we add two possibilities. How many Calabi-Yauspaces are there for $d = 3$? Despite considerable effort by mathematicians in recent years, at this point in time it is not known if the number is finite or even if there are any bounds on the Euler characteristic! There are certainly many methods of building Calabi-Yau manifolds and all constructions which magically produced a K3 surface (or occasionally a complex two-torus) for $d = 2$ tend to produce something different in each case when applied to $d = 3$. There are some constructions which can systematically produce a very large number of Calabi-Yau manifolds. Perhaps the most general construction studied so far is that of complete

intersections in toric varieties [38]. There are known Calabi-Yau manifolds which cannot be built this way however [39]. See [17] for an introduction to a few of these techniques.

There are some simple statements that can be made. Firstly all Calabi-Yau spaces with $h^{2,0} \neq 0$ are the complex 3-torus or a K3 surface times a one-torus or some orbifold thereof [40]. We will ignore such possibilities here although it should not be difficult using the results of the preceding sections to analyze the moduli spaces completely for these examples. Making the assumption $h^{2,0} = 0$ removes some of the complications that occured in the previous section. The effect of the complex structure on the Hodge decomposition $H^2(X, \mathbb{C}) \cong H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$ will now be trivial since $H^2(X, \mathbb{C}) \cong H^{1,1}$. This allows us to factorize (at least locally away from the boundary) the moduli space into deformations of complex structure and deformations of the Kähler form.

Unfortunately we pay dearly for this simplification. It was the $h^{2,0} = 1$ property of the K3 surface that allowed us to build the $N{=}4$ superconformal algebra that in turn controlled the perturbation theory for the non-linear σ-model. Surprisingly one receives no correction to the β function for the metric at two and three loop with only $N{=}2$ supersymmetry but at four loops we obtain an extra correction [41] for the metric:

$$\beta_{i\bar{j}} = -\frac{1}{2\pi} R_{i\bar{j}} - \frac{4\zeta(3)}{3(4\pi)^4} \partial_i \partial_{\bar{j}} \left[R_{k\bar{l}m\bar{n}} R^{p\bar{l}q\bar{n}} R_p{}^k{}_q{}^m - R_{k\bar{l}m\bar{n}} R^{m\bar{n}p\bar{q}} R_{p\bar{q}}{}^{k\bar{l}} \right] + \ldots \tag{50}$$

Thus we no longer have a Ricci-flat metric as a solution. Actually this doesn't really matter since we never actually found the metric in the last section, we just needed Yau's theorem. We now need some modified version of the theorem to say that given X with a complex structure and a point in the Kähler cone of X, there will be a unique solution of (50) (at least in some large radius limit) such that the Kähler form is of the desired cohomology class. We will assume that this is the case for convenience of argument. Ultimately we do not really care about the metric and this point will be irrelevant.

More worryingly, we now have instanton effects which are non-trivial. As we shall see this will cause severe problems with a metric-based point of view but fortunately we have the algebrao-geometric methods of section 3 to rescue us.

The two factors of the moduli space are best viewed using topological field theory. We refer to M. Bershadsky's lectures in this volume for more details regarding the following summary. The main point is that a topological field theory has only a few observables and correlation functions which are fairly simple to calculate (at least compared to non-topological field theories). We can "twist" an $N{=}2$ superconformal field theory in two different ways to obtain two distinct topological field theories. This is achieved via the transformation

$$\begin{aligned} T(z) &\to T(z) \pm \tfrac{1}{2}\partial J(z), \\ \bar{T}(\bar{z}) &\to \bar{T}(\bar{z}) + \tfrac{1}{2}\bar{\partial}\bar{J}(\bar{z}). \end{aligned} \tag{51}$$

These two different twistings give rise to the "A-model" and the "B-model" as defined in [42]. Only the (anti)chiral superfields of the $N{=}2$ theory appear in the A and B models. In

particular, only the (c,c) and (a,a) fields appear in one and the (c,a) and (a,c) fields appear in the other (again see [13] for an explanation of this notation).

When analyzed in terms of a non-linear σ-model the correlation functions of the fields in the A-model depend only upon the Kähler form and B-term of X and the correlation functions of the B-model depend only upon the complex structure [42]. Thus our factors of the moduli space of N=2 theories coincide with the moduli spaces of the topological field theories obtained by twisting. Thus, although considerable information is lost when one twists an N=2 theory into a topological field theory, so long as we consider both twists, we do not lose any information about the structure of the moduli space.

Let us briefly review each moduli space. The moduli space of the A-model $\mathscr{M}_A$ encodes information concerning the Kähler form and B-term of X. This occurs because of instantons. A solution to the equations of motion of the A-model is a holomorphic map from the world-sheet, Σ, to X. The constant map mapping Σ to a point in X is the trivial case. Any other solution is an instanton. If this map is one-to-one then the image of Σ in X is an algebraic curve. In the case $d = 3$ it turns out that three-point functions between interesting observables are only non-zero when the world-sheet is genus zero. The curves in question are therefore rational. The remaining instantons unaccounted for are the non-trivial many-to-one maps. These are multiple covers of rational curves (for which the interested reader is referred to [43]).

In the case that an instanton corresponds to a rational curve, C, in X, the value of the action is $S = -2\pi i \int_C (B + iJ)$. Let us chose a basis, e_k, for $H^2(X, \mathbb{Z})$. Using our assumption that $h^{2,0} = 0$ we may then make the following definitions

$$B + iJ = \sum_k (B + iJ)_k \, e_k,$$

$$q_k = e^{2\pi i(B+iJ)_k}, \quad k = 1, \ldots, h^{1,1}(X). \tag{52}$$

It then follows that any correlation function will be a power series in the variables q_k.

Note that we can now make more precise our rather vague notion of "large radius limit" for the non-linear σ-model. As each component of the Kähler form, J_k, tends to infinity, we see that $|q_k| \to 0$. Thus any power series is more likely to converge. We will define the large radius limit to be the limit in which $J_k \to \infty$ for all k. Thus, not only does X become a space of infinite volume but each algebraic subspace will also become infinitely large. We will consider this large radius limit $q_k = 0$ to be a point in the moduli space. With this definition we may hope that the correlation functions of the A-model will converge in some non-zero region around this large radius limit point.

The B-model and its moduli space $\mathscr{M}_B$ are in many ways much simpler. The only solution to the equations of motion in this case are the constant maps. Therefore we are free from instantons. This makes the non-linear σ-model very simple to analyze for information concerning observables which appear in the B-model. Essentially the three-point functions calculated at tree-level in the large radius limit will be exact. Actually this fact should

have been clear once we noted the factorization of the moduli space into $\mathscr{M}_A \times \mathscr{M}_B$. If the correlation functions of the B-model depend only on our position with $\mathscr{M}_B$, we can go as close to the large radius limit in $\mathscr{M}_A$ as we please thus taking the classical limit. This has a profound consequence:

> *The moduli space of B-models, $\mathscr{M}_B$, associated to a Calabi-Yau space X is isomorphic to the classical moduli space of complex structures on X.*

Thus the fact that the left-hand side of figure 2 is the moduli space of complex structures for an elliptic curve was not an artifact of the simplicity of this model. This kind of behaviour persists in higher dimensions.

In the case of the complex one torus we just had to copy $\mathscr{M}_B$ to obtain $\mathscr{M}_A$. In general this situation is not quite so straight-forward. Let us consider a non-linear σ-model with target space X. We may then associate to this a conformal field theory and thus a moduli space if X is a Calabi-Yau space. It may (or may not) be the case that there is another Calabi-Yau space Y which yields exactly the same conformal field theory as that given by X except that we exchange chiral rings (c,c)↔(a,c) and (c,a)↔(a,a). In this case Y is the "mirror" of X (again the reader is referred to [13].) If this is the case then clearly $\mathscr{M}_A(X) \cong \mathscr{M}_B(Y)$ and $\mathscr{M}_A(Y) \cong \mathscr{M}_B(X)$. For the cases $d < 3$ the condition of being a Calabi-Yau is so constraining that X and Y are topologically the same. Thus the mirror map appears as an automorphism on the moduli space. For $d \geq 3$ it is usually the case that X and Y are topologically distinct (or even that Y does not exists as Calabi-Yau manifold).

Assuming Y exists and we can identify it, we may thus calculate the moduli space of N=2 theories associated to X by two complex structure moduli space calculations — one for $\mathscr{M}_B(X)$ and one for $\mathscr{M}_A(X) \cong \mathscr{M}_B(Y)$. The moduli space is then generically $(\mathscr{M}_A(X) \times \mathscr{M}_B(X))/\mathbb{Z}_2$ where the final $\mathbb{Z}_2$ quotient corresponds to complex conjugation of X and changing the sign of B.

5.2   The Gauged Linear σ-model

Before we proceed further to look at a simple example, we first introduce another field theory associated to the target space X. The reason for this short diversion should become clear soon. This will be Witten's linear σ-model. We refer the reader to the original paper [2] for more details concerning the remainder of this subsection (see also J. Distler's lectures in this volume). The basic idea is that although the conformally invariant σ-model for X is very complicated we may consider a much simpler σ-model with a larger target space containing X such that the fields are constrained to live in X in some limit. The latter σ-model (which we will refer to as the *linear* σ-model) shares many properties of the nonlinear σ-model which has X as the genuine target space.

Before writing down this linear σ-model we need to specify our conventions for superspace. To fit in with our earlier description of the N=2 superalgebra we denote by $(z, \theta^\pm)$ our

coordinates for the left sector and $(\bar{z}, \bar{\theta}^{\pm})$ for the right sector.[4] Let us introduce a set of charged (c,c) superfields Φ_i and a set of neutral (c,a) superfields Σ_l in a supersymmetric gauge theory. That is,

$$\mathscr{D}_+ \Phi_i = \overline{\mathscr{D}_+} \Phi_i = \mathscr{D}_+ \Sigma_l = \overline{\mathscr{D}_-} \Sigma_l = 0, \tag{53}$$

where $\mathscr{D}$ is the gauge covariant superderivative. For our purposes we consider the case where the gauge group is $U(1)^s$ and $l = 1, \ldots, s$. Each Σ_l may be considered as the supersymmetric version of the field strength for the lth $U(1)$ factor [2]. The gauge fields for this theory live in vector superfields V_l. We may write

$$\Phi_i = \exp(\sum_{l=1}^{s} Q_i^{(l)} V_l) \Phi_i^0, \tag{54}$$

where $Q_i^{(l)}$ is the charge of Φ_i with respect to the lth $U(1)$. In this case Φ_i^0 is a (c,c) superfield with respect to the standard (rather than gauge covariant) superderivatives. We may then expand Φ_i^0 as

$$\Phi_i^0 = \phi_i + \psi\theta^- + \bar{\psi}\bar{\theta}^- + F_i\theta^-\bar{\theta}^- + \ldots \tag{55}$$

Note that for a rigid $U(1)$ rotation, $\Phi_i \to e^{i\alpha}\Phi_i$, the vector superfield is invariant and so $\phi_i \to e^{i\alpha}\phi_i$. The details of the vector superfield are rather messy. The "D-term" appears in the form $D_l \theta^+\theta^-\bar{\theta}^+\bar{\theta}^-$ in the expansion. A vector superfield in four dimensions has a real vector boson as its lowest component. When we dimensionally reduce to two dimensions we thus obtain a two-dimensional vector, which we denote v_l and two real scalars which we combine into a complex scalar σ_l.

The action to be considered is

$$S = \int \sum_{i=1}^{N} \left(\bar{\Phi}_i \Phi_i\right) d^2z d^4\theta - \frac{1}{4e^2} \int \sum_{l=1}^{s} \left(\bar{\Sigma}_l \Sigma_l\right) d^2z d^4\theta$$
$$- \int W(\Phi_i) d^2z d^2\theta^- + \frac{i}{2\sqrt{2}} \sum_{l=1}^{s} (\beta + ir)_l \int \Sigma_l d^2z d\theta^+ d\bar{\theta}^- + \text{h.c.}, \tag{56}$$

where W is a neutral holomorphic function and r_l and β_l are real numbers.

The equations of motion for this model together with the condition that we lie in the

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[4]Note that this differs from [2].

fixed point set of the fermionic symmetries[5] give

$$D_l = -e^2 \left(\sum_{i=1}^{N} Q_i^{(l)} |\phi_i|^2 - r_l \right) = -\mathbf{F}_l$$

$$F_i = \frac{\partial W}{\partial \phi_i} = 0 \qquad (57)$$

$$\bar{D} \phi_i = 0$$

$$Q_i^{(l)} \phi_i \sigma_l = 0$$

The solutions to these equations are governed by the topological invariant

$$\int \mathbf{F}_l \, d^2 z = -2\pi n_l, \qquad (58)$$

where $\mathbf{F}_l$ is the field strength of v_l. This is topological since it may be considered to be the integral of the first Chern class over a space. Assuming single valuedness of sections of the bundle of which $\mathbf{F}$ is the curvature would force n to be integer. This is the two-dimensional analogue of the familiar second Chern class $\int \mathrm{tr}(F \wedge F) \, d^4 x$ term that appears in four-dimensional Yang-Mills. One may also show [2] that $\sum_l r_l n_l \geq 0$.

The first and third equation of (57) may be combined to complexify the $U(1)^s$ gauge group to $(\mathbb{C}^*)^s$ [2, 44]. ($\mathbb{C}^*$ is the group, under multiplication, of nonzero complex numbers.) We need only work at tree-level for the purposes of these lectures and thus let us assume that Σ is of genus zero. The third equation in (57) may then be used to tell us that ϕ_i is a holomorphic section of the line bundle $\mathcal{O}(\sum_l Q_i^{(l)} n_l)$. The only holomorphic section of $\mathcal{O}(m)$, where m is negative, is the trivial zero section and so we have

$$\sum_{l=0}^{s} Q_i^{(l)} n_l < 0 \quad \Rightarrow \quad \phi_i = 0. \qquad (59)$$

This will force some, but in general not all, of the fields ϕ_i to vanish. Therefore, generically, the last equation in (57) forces σ_l to vanish.

For the classical vacuum we need to find the classical potential. This is given by [2, 45]

$$U = \frac{1}{2e^2} \sum_{l=1}^{s} D_l^2 + \sum_{i=1}^{N} |F_i|^2 + 2 \sum_{k,l=1}^{s} \sum_{i=1}^{N} \bar{\sigma}_k \sigma_l Q_i^{(k)} Q_i^{(l)} |\phi_i|^2, \qquad (60)$$

where the equations of motion set the auxiliary fields as in (57).

There is one very useful fact which we should immediately note concerning the parameter β_l. This appears as the coefficient of the term (58). β_l behaves just like the "theta" in four-dimensional Yang-Mills theory, which only affects the field theory via instantons and correlations are periodic in this variable. In particular, if n is an integer then $\beta_l \cong \beta_l + 1$.

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[5] This is where the topological field theory localizes [2].

5.3 An example

Further discussion of the case $d = 3$ and the linear σ-model is best illuminated by a specific example. From previous sections we have already discussed two simple methods of construction one might use to build a Calabi-Yau threefold. The first method from section 4 would be to take a complex three-torus and divide out by a subgroup of $SU(3)$ and blow-up the resulting singularities. The other method would be to take a complex projective space $\mathbb{P}^4$ with homogeneous coordinates $[x_0, \ldots, x_4]$ and define a hypersurface, X, by the vanishing of a generic polynomial of homogeneous degree 5 in x_i. Both of these examples appeared in the original paper on Calabi-Yau manifolds in string theory [46].

The latter space above, the "quintic threefold", is particularly attractive because of the simplicity of $\mathcal{M}_A$. The moduli space $\mathcal{M}_A(X)$ of this Calabi-Yau space was computed first in [47]. Taking a hyperplane $\mathbb{P}^3$ and intersecting with the quintic threefold we obtain a divisor. (Since $d = 3$, divisors will now have complex dimension 2.) This divisor generates $H_4(X, \mathbb{Z})$ and thus there is a two-form, e, associated (by the same "dual of a dual" construction of section 4) to this divisor that generates $H^2(X, \mathbb{Z})$. Thus the dimension of $H^2(X)$ is one. The full Hodge diamond of this space is

$$
h^{p,q}(X) = \begin{matrix}
& & & 1 & & & \\
& & 0 & & 0 & & \\
& 0 & & 1 & & 0 & \\
1 & & 101 & & 101 & & 1 \\
& 0 & & 1 & & 0 & \\
& & 0 & & 0 & & \\
& & & 1 & & &
\end{matrix} \, . \tag{61}
$$

Therefore $\mathcal{M}_A$ is has complex dimension 1 and $\mathcal{M}_B$ has dimension 101. Before discussing Y and $\mathcal{M}_B(Y)$ let us explore the linear σ-model of X. For our example we set $s = 1$ and $N = 6$. Write $\Phi_1, \ldots, \Phi_5$ as $X_0, \ldots, X_4$ and Φ_6 as P. Also set

$$
\begin{aligned}
W &= P.f(X_i) \\
&= P(X_0^5 + X_1^5 + X_2^5 + X_3^5 + X_4^5 + \ldots),
\end{aligned} \tag{62}
$$

where ... represents other terms (with arbitrary coefficients) of degree 5 in the X_i's. For W to be neutral we set the $U(1)$ charge of each X_i to be $+1$ and the charge of P to be -5. We will also use the letters x_i and p to denote the complex bosons which are the lowest respective components of the associated (c,c) superfields in the sense of (55). The classical potential energy becomes

$$
U = \left[|x_0|^2 + \ldots + |x_4|^2 - 5|p|^2 - r \right]^2 + |f(x_i)|^2 + |p|^2 \sum_{i=0}^4 \left| \frac{\partial f}{\partial x_i} \right|^2 + 2|\sigma|^2 \left(\sum_{i=0}^4 |x_i|^2 + 25|p|^2 \right) . \tag{63}
$$

29

We also have from (57) that x_i are sections of $\mathcal{O}(n)$ and that p is a section of $\mathcal{O}(-5n)$. In addition, the fields ϕ_i satisfy the constraint $f(x_i) = 0$.

Suppose for the time being that $r \gg 0$. Since $nr \geq 0$, we have $n \geq 0$. If $n > 0$ then p must be zero by (57). Clearly U is non-negative. Let us try to solve for $U = 0$ — this will be the $n = 0$ case. To obtain zero for the first term in (63) we must have that at least one of the x_i's is non-vanishing. The last term then forces σ to vanish. Assuming the transversality condition holds, i.e., that (28) is only satisfied when all the x_i's are zero, then p is forced to vanish by (57).

The classical vacuum, i.e. $U = 0$, is thus parametrized by x_i subject to the constraints

$$|x_0|^2 + \ldots + |x_4|^2 = r$$
$$f(x_i) = 0. \tag{64}$$

We also have the $U(1)$-action $x_i \to e^{i\alpha} x_i$. The first equation in (64) gives the sphere S^9. Dividing this by the $U(1)$ action gives $S^9/S^1 \cong \mathbb{P}^4$. The easiest way to see this latter isomorphism is to build $\mathbb{P}^4$ in stages as follows:

1. Take the space $\mathbb{C}^5$ with coordinates $(x_0, \ldots, x_4)$ and remove the origin $\{O\}$.

2. Divide any vector in $\mathbb{C}^5 - \{O\}$ by its length. This retracts $\mathbb{C}^5 - \{O\}$ onto the sphere S^9. It may be considered as the quotient $(\mathbb{C}^5 - \{O\})/\mathbb{R}_+ \cong S^9$ where $\mathbb{R}_+$ is the group of positive real numbers.

3. Since $\mathbb{P}^4 \cong (\mathbb{C}^5 - \{O\})/\mathbb{C}^*$ and $\mathbb{C}^* \cong S^1 \times \mathbb{R}^+$, we have $\mathbb{P}^4 \cong S^9/S^1$ where the S^1 acts as $x_i \to e^{i\alpha} x_i$.

Finally, therefore, our target space consists of the hypersurface $f(x_i) = 0$ in the projective space $\mathbb{P}^4$ exactly as desired. It should be noted that this manipulation extending a $U(1)$ quotient to a $\mathbb{C}^*$ quotient is essentially equivalent to the procedure referred to earlier for identifying ϕ_i as sections of holomorphic bundles.

Deforming the quintic equation $f(x_i)$ will produce deformations of complex structure of X. More interestingly, changing r will change the radius of S^9 and thus the overall scale of X. This is equivalent to deforming the single parameter governing the Kähler form of X. Thus, r appears to be a degree of freedom similar to J. It is also tempting to associate β with B since they both live on a circle and combine with r and J respectively to form natural complex parameters. Before we make such a bold identification however we must bare in mind that we have not really built the desired non-linear σ-model with target space X.

The fields x_i originally span the space $\mathbb{C}^5$. The metric on this space is the trivial one (implicitly present in the first term in (56)). The metric on $\mathbb{P}^4$ is inherited from this original metric by "symplectic reduction" [48]. This produces the "Fubini-Study" metric on $\mathbb{P}^4$. The metric on X is therefore the restriction of the Fubini-Study metric. Such a metric is not

Ricci flat and so cannot satisfy (50) in the large radius limit. Actually we have not even the correct degrees of freedom in the linear σ-model. In the desired nonlinear σ-model the fields are completely constrained to lie in the target space X. While we appear to have the correct behaviour for the low-energy behaviour in the linear σ-model one should expect extra massive fields which are not confined to live in X. Therefore, in order to obtain the correct degrees of freedom we should integrate out such massive states. Such a procedure may well affect the value of the parameters β and r.

The correlation functions of the A-model depended on $B + iJ$. This dependence came from instanton effects. To compare the quantities $B + iJ$ and $\beta + ir$ it should be useful to look at instantons in the linear σ-model. The interested reader is referred to [45] for more features of these instantons. The instantons are solutions to the equations (57) but will not necessarily satisfy $U = 0$. In particular we have that x_i are sections of $\mathcal{O}(n)$. This forces n to be an integer and thus $\beta \cong \beta + 1$ as explained in the previous section.

Now let us compare this linear σ-model instanton with the A-model instantons.[6] An A-model instanton is a holomorphic map from Σ into X. Homogeneity requires n to be the same for each x_i. Now consider a hyperplane $\mathbb{P}^3 \subset \mathbb{P}^4$. The image of Σ under the map given by x_i will intersect this hyperplane n times if the map is suitably generic. This is the *degree* of the map.

Thus far the linear σ-model instantons and the A-model instantons appear identical. There is a difference however. For the A-model, the quantities x_i are homogeneous coordinates of $\mathbb{P}^4$ and cannot simultaneously vanish. The x_i's of the linear σ-model are just sections of $\mathcal{O}(n)$ and therefore not so constrained. Suppose we have an instanton in the linear σ-model where all of the fields x_i vanish at a point $z_0 \in \Sigma$ and let us assume that $n = 1$. The equations (57) and (58) dictate that

$$\int_\Sigma \left(\sum_{i=0}^4 |x_i|^2 - r \right) d^2z = \frac{2\pi}{e^2}. \tag{65}$$

Since $r \gg 0$, the value of $\sum_i |x_i|^2$ must rise rapidly as one moves away from z_0. In fact, the region on the world sheet around z_0 where $\sum_i |x_i|^2$ is not roughly equal to r must have an area the order of $1/e^2r$. That is, this instanton appears as a small lump around z_0.

We see then that the linear σ-model contains all the instantons of A-model and in addition some small-scale instantons which shrink down to points on the world-sheet in the limit $r \to \infty$. In order to translate between the coordinate r and the coordinate J on the respective moduli spaces we need to take into account the effects of these small-scale instantons.

In addition to this non-perturbative correction to r we should also consider loops in the perturbation theory. The super-renormalizability of this gauge theory in two dimensions makes such an analysis rather straight-forward. We need to consider one-loop diagrams such

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[6]The author wishes to thank R. Plesser for discussions on this point which has overlap with [45].

as the tadpole given by

(66)

This results in a shift in $\beta_l + ir_l$ which we will denote Δ_l. The value of this shift will vary depending upon our position in the moduli space but should be expected to be finite.

Let us write $t_l = (B + iJ)_l$ and $\tau_l = (\beta + ir)_l$, where r is the *uncorrected* parameter in the linear σ-model (i.e., we have not taken into account the one-loop effects). To calculate how t_l is related to τ_l we combine the one-loop and instanton corrections. This amounts to

$$t_l = \tau_l + \Delta_l + \sum_{m=1}^{s} K_m e^{2\pi i \tau_m} + \dots \qquad (67)$$

where K_m represents the first-order effect from the small-scale instantons and ... represents the higher orders. Let us introduce a variable z_l as an analogy to the variable q_l of (52):

$$z_l = \pm \exp\left\{2\pi i \tau_l\right\}. \qquad (68)$$

The above sign ambiguity is a problem that always occurs in the problem of trying to find the relationship between q and z. The idea is that it appears that one can choose a sign so that [49]

$$q_l = z_l(1 + C_l), \qquad (69)$$

where C_l is a power series in $z_1, z_2, \dots$ with no constant term. That is, we assume that the Δ term corrects τ_l by 0 or π. One can determine this explicitly by counting rational curves in the example being studied as was done in [47] for example. This conjecture has yet to be proven in general. Standard renormalization arguments in $N=2$ theories guarantee that C_l is a holomorphic function.

Thus far the reader may wonder what we have achieved by rephrasing things in terms of the linear σ-model. The answer is that we may now probe the moduli space a long way from the large radius limit at least in some sense — the linear σ-model may be analyzed away from the limit $r \to \infty$, i.e., $|z| \to 0$.

As an extreme example let us consider the case $r \ll 0$. Now $n \leq 0$ and the fields x_i are forced to be zero (using the transversality condition when $n = 0$). For the classical vacuum we first look at the vanishing of U in (63). This will force p to be nonvanishing. This in turn forces σ to vanish. The equation $f(x_i) = 0$ is now trivially satisfied. The value of r fixes $|p|$ and the $U(1)$ gauge symmetry may be used to fix the phase of p. Thus all the expectation values of the fields are fixed — the classical target space is a point! Actually, to be more precise, we have a Landau-Ginzburg orbifold theory. The fields x_i may have a zero vacuum expectation value but their quantum fluctuations are massless and governed by the

superpotential W which is of degree 5 in the fields x_i. We also need to note that the field p has charge -5 while the charge of the x_i's is 1. Therefore, when the phase of p was fixed by using the symmetry $\phi \rightarrow \exp(2\pi i\alpha Q_i)\phi$, we still have a residual $\mathbb{Z}_5$ symmetry

$$g: x_i \mapsto e^{2\pi i/5}x_i. \tag{70}$$

Therefore, the fields x_i effectively live in $\mathbb{C}^5/\mathbb{Z}_5$ and our theory is an orbifold.

Again this theory has instantons. Now p will be nontrivial and a section of $\mathcal{O}(-5n)$. This means that $n \in \mathbb{Z}/5$ but that n need not be integral. Let us consider the instanton given by $n = -\frac{1}{5}$. p is a section of $\mathcal{O}(1)$ and will have one zero on Σ. Let us denote this point on the world-sheet by z_0. Since

$$\int_\Sigma e^2(-5|p|^2 - r)\, d^2z = \int_\Sigma \mathbf{F} d^2z = \tfrac{1}{5}, \tag{71}$$

we must have that the value of $|p|^2$ rises quickly to $|r/5|$ outside a patch of area of order $-1/e^2r$ around this zero on the world-sheet. Thus the interesting part of this instanton configuration is confined to a small lump around this single zero. The fields x_i live in the bundle $\mathcal{O}(-1/5)$ and will not be single valued. In particular, the field p will pick up a phase 2π with respect to monodromy around z_0 whereas x_i will pick up a phase of $-2\pi/5$ around the same point $z_0 \in \Sigma$. This is precisely a twist-field configuration where the map x_i is taking the point z_0 on the world-sheet to the origin of the target $\mathbb{C}^5/\mathbb{Z}_5$.

Thus by varying the value of r, we may switch between a target space of a Calabi-Yau hypersurface in $\mathbb{P}^4$ (for which the only massless modes lie within that space) and a target space which is a point with massless Landau-Ginzburg-type massless fluctuations about it. Each of these theories has instantons and in each case the action of the instantons goes as $|r|$ so that the instanton effects become negligible in the large $|r|$ limit. That is, we have "exactly" a Calabi-Yau theory for $r = \infty$ (in an infinitely large target space) and "exactly" a Landau-Ginzburg orbifold theory for $r = -\infty$.

We should ask if anything peculiar happens for a finite value of r as we change between these two "phases". The only special value at which something nasty happens occurs classically for $r = 0$. At this value, all the fields x_i and p may vanish and then σ may take on any value. Any analysis which works for the theory at generic r values should be expected to contain divergences when r takes on this value and σ becomes massless. Actually we have to be a little more careful than this (and we refer the reader again to [2] for more detail). Taking quantum effects properly into account, the minimum energy density for a state at large σ goes roughly as $r^2 + \beta^2$. Thus we have a singularity at $\beta + ir = 0$. This is the true corrected value for β and r at the singularity so we need to compute Δ at this point in the moduli space to find z. This may be calculated by assuming that σ is large and is given

purely by the diagram (66). The result in the case that $s = 1$ is that [45]

$$
\begin{aligned}
\Delta &= -\frac{1}{2\pi} \sum_{i=1}^{N} Q_i \log Q_i \\
&= \frac{5}{2\pi} \log 5 - \tfrac{5}{2}i.
\end{aligned}
\tag{72}
$$

Thus $z = \pm 5^{-5}$ at the singularity. It turns out that we should choose $z = 5^{-5}$ to get the rational curve count correct [47].

To summarize thus far we have identified three special points in the moduli space. At $z = 0$ we have a Calabi-Yau target space of infinite radius. We also know that in the neighbourhood of $z = 0$ we may use (69) to relate this coordinate to the coordinate in the moduli space of A-models on the same Calabi-Yau target space. At $z = \infty$ the linear σ-model is equivalent to a Landau-Ginzburg orbifold and at $z = 5^{-5}$ the theory is singular.

Now let us address the question of $\mathscr{M}_B(Y)$ which we expect to be isomorphic to $\mathscr{M}_A(X)$. First let us carefully build Y. Analysis of the linear σ-model showed us that there is a Landau-Ginzburg orbifold theory in the moduli space. The same must be true for the theory of A-models (although at this point in the argument we don't know where it is located) and therefore $N{=}2$ conformal field theories. If we go to the correct point in $\mathscr{M}_B(X)$, the Landau-Ginzburg theory in question may be written as a tensor product of minimal $N{=}2$ models and thus the required Landau-Ginzburg orbifold is an orbifold of such a tensor product [50, 51, 52]. The resulting theory is a "Gepner Model" [53]. The process for finding the mirror of a Gepner model is well-understood [54] (see also [13]). The result is that the mirror is a certain orbifold of the original Gepner model. This may then be retranslated back into Landau-Ginzburg orbifold language.

The required result is as follows. If X_{Gep} is the Landau-Ginzburg theory defined by the superpotential[7]

$$
W = x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5
\tag{73}
$$

in $\mathbb{C}^5/\mathbb{Z}_5$ where $\mathbb{Z}_5$ is generated by (70), then the mirror theory Y_{Gep} is the Landau-Ginzburg orbifold theory with the same superpotential in the space $\mathbb{C}^5/(\mathbb{Z}_5)^4$, where $(\mathbb{Z}_5)^4$ is the group consisting of elements

$$
g\colon (x_0, x_1, \ldots, x_4) \mapsto (\alpha^{n_0} x_0, \alpha^{n_1} x_1, \ldots, \alpha^{n_4} x_4),
\tag{74}
$$

where α is a non-trivial fifth root of unity and n_i are integers such that the relation $\sum_i n_i = 0$ (mod 5) holds.

Now we need to look at $\mathscr{M}_B(Y)$. This is given by the moduli space of superpotentials. In the case of $\mathscr{M}_B(X)$ there was a 101 dimensional space of superpotentials, or, equivalently, a

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[7]From this point onwards we will use lower case to refer to chiral superfields as well as their lowest bosonic component. We hope the context makes it clear which is relevant.

101 dimensional space of complex structures on X. The $(\mathbb{Z}_5)^4$ orbifolding group will project most of these deformations out. As expected, we are left with one parameter. We may write f in the form

$$f = x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 - 5\psi x_0 x_1 x_2 x_3 x_4, \tag{75}$$

where ψ is the parameter. Thus, by construction, $\psi = 0$ is the Landau-Ginzburg orbifold point which is mirror to the Landau-Ginzburg orbifold point for X. Varying ψ will span the space $\mathscr{M}_B(Y)$. At this point Y is a Landau-Ginzburg orbifold. As we found with X however, the space of $\mathscr{M}_A(Y)$ will be such that Y may be pictured as a smooth Calabi-Yau manifold in part of the moduli space of A-models. Since we are only concerned with B-model data for Y we may take Y to be this smooth Calabi-Yau manifold. As a matter of fact there will be many smooth Calabi-Yau manifolds which may be used to represent Y. This issue will be explained further in section 6.

Since we now have a global description of $\mathscr{M}_B(Y)$ and of linear σ-models on X, it is natural to try to map them to each other. First we should pick good coordinates on each moduli space such that two different values of the coordinate do not correspond to the same point. The identification $\beta \to \beta+1$ in the neighbourhood of $z = 0$ makes z a good coordinate at least in the vicinity of the large radius Calabi-Yau limit. Around the Landau-Ginzburg point, the instanton number n need only be an element of $\mathbb{Z}/5$ and so it would appear that the identification should be $\beta \to \beta + 5$. Actually there is a $\mathbb{Z}_5$ symmetry around the point $z = \infty$ which identifies theories such that we restore $\beta \to \beta + 1$. As indicated earlier, the instantons with non-integer n may be thought of as twist fields. A twist field ξ_n then transforms under this $\mathbb{Z}_5$ symmetry as

$$g \colon \xi_n \to e^{2\pi i n} \xi_n. \tag{76}$$

Therefore z is a good coordinate locally around $z = \infty$. Picturing the moduli space as $\mathbb{P}^1$ with z the usual affine coordinate, we have done enough to show that z is a good coordinate everywhere (together with the point $z = \infty$).

The $\mathbb{Z}_5$ transformation generated by

$$g \colon (x_0, x_1, \ldots, x_4) \mapsto (\alpha x_0, x_1, \ldots, x_4) \tag{77}$$

can be made a symmetry of (75) by extending it so that $g \colon \psi \to \alpha^{-1}\psi$. Therefore ψ is not a good coordinate. This symmetry of ψ is the only one induced by symmetries of the equation (75). This means that ψ^5 is a good coordinate.

We now wish to map these two $\mathbb{P}^1$ moduli spaces to each other in a one to one manner. The pervading $N{=}2$ structure present means that we expect this map to be holomorphic. This forces

$$z = \frac{a\psi^5 + b}{c\psi^5 + d} \tag{78}$$

where a, b, c, d are complex numbers such that $ad - bc = 1$. We may thus fix this map by identifying three points. We have already identified $z = \infty$ and $\psi = 0$ as the Landau-Ginzburg orbifold point. We know that $z = 5^{-5}$ corresponds to a singular theory. The equation (75) satisfies (28) when $\psi^5 = 1$. This therefore provides a natural candidate for the singular point in $\mathcal{M}_B(Y)$. Actually, if one attempts to calculate correlation functions using the "chiral ring" of [50] then it is precisely when (28) is satisfied that these calculations become badly defined. Thus we map $z = 5^{-5}$ to $\psi^5 = 1$. For our last point we pick the large radius Calabi-Yau point $z = 0$. Clearly it would be unnatural for this point to be anything other than $\psi = \infty$. To check this however one may put the Zamolodchikov metric on $\mathcal{M}_A(X)$ and $\mathcal{M}_B(Y)$. The point $z = 0$ maps to $q = 0$ in $\mathcal{M}_A(X)$ which corresponds to the large radius limit and is thus an infinite distance away from other points on $\mathbb{P}^1$. The same is true for $\psi = \infty$ on $\mathcal{M}_B(Y)$ [47]. We have thus fixed

$$z = (5\psi)^{-5}. \tag{79}$$

It is amazing just how simple (79) is. This is the justification of our using the linear σ-model. While the map between $\mathcal{M}_A(X)$ (in terms of q) and $\mathcal{M}_B(Y)$ (in terms of ψ) is rather complicated, there is a very simple relation between the linear σ-model version of $\mathcal{M}_A(X)$ (in terms of z) and $\mathcal{M}_B(Y)$. At this point it is very tempting to end our analysis of the moduli spaces since in many ways we have all the information we need concerning the moduli space. The only problem is that we have described $\mathcal{M}_A(X)$ in terms of the coordinate z rather than q. Recall that q was derived from the Kähler form and thus ultimately differential geometry. It appears that z is the coordinate that appears more naturally in string theory and perhaps we should rewrite general relativity in terms of degrees of freedom naturally expressed in terms of this parameter rather than q. Rather than attempting such an ambitious problem we will submit to the present conventions for describing space-time and try to reparametrize our moduli space in terms of q.

Actually the analysis of section 3.2 allows us to do this without much more work. Recall that we found $(B + iJ)_l$ in terms of a ratio of periods on the mirror, in this case Y. One may use the local geometry of the moduli space $N{=}2$ theories to show that this is also the case for any d [55]. Therefore we just need the Picard-Fuchs equation for these periods to find q as a function of z. The desired equation for the periods ϖ is [47]

$$\left(z\frac{d}{dz}\right)^4 f - z\left(z\frac{d}{dz} + \tfrac{1}{5}\right)\ldots\left(z\frac{d}{dz} + \tfrac{4}{5}\right) f = 0. \tag{80}$$

The four solutions of this hypergeometric differential equation are characterized by their monodromy around $z = 0$, which goes as $1, \log z, (\log z)^2, (\log z)^3$ respectively. The powerful constraint (69) then determines the answer uniquely to be

$$q = \exp\left(\frac{\varpi_1}{\varpi_0}\right), \tag{81}$$

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where

$$\varpi_0 = {}_4F_3(\tfrac{1}{5}, \tfrac{2}{5}, \tfrac{3}{5}, \tfrac{4}{5}; 1, 1, 1; 5^5 z)$$

$$= \sum_{n=0}^{\infty} \frac{(5n)!}{(n!)^5} z^n \tag{82}$$

$$\varpi_1 = \varpi_0. \log z + 5 \sum_{n=1}^{\infty} \frac{(5n)!}{(n!)^5} z^n \left[\Psi(1+5n) - \Psi(1+n) \right],$$

where $\Psi(x)$ is the digamma function defined as the derivative of $\log \Gamma(x)$. These periods may also be found directly from a smooth model of Y without going via the Picard-Fuchs equation [19].

We may now expand q as

$$q = z + 770z^2 + 1014275z^3 + \dots \tag{83}$$

which converges for $|z| \leq 5^{-5}$. Thus, by using mirror symmetry we have been able to determine the precise form of (69). It should come as no surprise that this series has a radius of convergence equal to 5^{-5} since the singular theory lies at $z = 5^{-5}$. At the point $z = 5^{-5}$ the series just manages to converge and we obtain

$$(B + iJ)_1 \approx 1.2056i. \tag{84}$$

We denote this value of J_1 by J_0. If we have a Calabi-Yau manifold diffeomorphic to the quintic hypersurface and if the size of it is such that the area of a curve which generates $H_2(X)$ is larger than about $1.2(4\pi^2\alpha')$ then we can place precisely where our model lies in the moduli space of $N{=}2$ theories by using (83).

We have not yet accounted for the complete moduli space of these theories however. What about the region where $|z| > 5^{-5}$? At this point the series fails to converge. If we try to compute correlation functions using the nonlinear σ-model for a point in this region of the moduli space we will also find that the instanton corrections form a divergent series too. If we were to be pragmatic at this point we should say that the nonlinear σ-model is not such a good picture for any theory in the region $|z| > 5^{-5}$ and that some other picture is more appropriate. Indeed, we already know what the other picture is — we should interpret such theories as Landau-Ginzburg orbifolds corrected by instantons. Actually, to use more conventional conformal field theory language, these instantons are twist fields. One may deform a conformal field theory orbifold by a twist field which is also a truly marginal operator. Although the resulting theory will not generically be an orbifold, one may compute correlation functions as a power series in terms of the fields in the original orbifold theory. This will be a power series in the coupling to the marginal operator in the usual way in conformal perturbation theory. One will find that such a series will converge when $|z| > 5^{-5}$.

For any point (except when $|z| = 5^{-5}$) in the moduli space $\mathscr{M}_A(X)$ we may therefore associate an effective target space theory. In one region we have a Calabi-Yau manifold with

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instanton corrections and in the other region we a Landau-Ginzburg orbifold perturbed by twisted marginal operators. Both types of theories have correlators in the form of convergent power series. The regions do not overlap.

This leads to the "phase" picture of the moduli space [2, 56]. We do not have some universal picture for a target space X associated to $\mathscr{M}_A(X)$. It is necessary to have a set of target space descriptions which are applicable in various regions in $\mathscr{M}_A(X)$.

The attentive reader may have realized that, because we have holomorphic functions, we should be able to analytically continue our series beyond their radius of convergence. Certainly this is possible as we now explain. One can therefore begin to interpret one phase of the moduli space in terms of other phases. The more cautious reader might declare such a process to be unnatural however. Anyway, let us see what happens when we extend the Calabi-Yau picture into the Landau-Ginzburg region.

To analytically continue we first need to apply branch cuts. The branch points are clearly $z = 0, 5^{-5}, \infty$. In our identification of the strip $(B + iJ)_1$, where $0 \leq B_1 < 1, J_1 \geq J_0$, with the hemisphere $|z| \leq 5^{-5}$ we have already cut from $z = 0$ to $z = 5^{-5}$. We require another cut emanating from $z = \infty$. To avoid changing the hemisphere around $z = 0$ which is already in the desired form, we cut form $z = \infty$ to $z = 5^{-5}$. This latter cut may be performed by returning to our variable ψ and imposing

$$-\frac{2\pi}{5} < \arg \psi < 0. \tag{85}$$

The same method of Barnes integrals as was used in section 3.2 then yields

$$(B + iJ)_1 = \tfrac{1}{2} + \tfrac{i}{2} \left\{ \cot \tfrac{\pi}{5} + \frac{\Gamma^4(\tfrac{4}{5})\Gamma(\tfrac{2}{5})}{\Gamma(\tfrac{1}{5})\Gamma^4(\tfrac{3}{5})}(\cot \tfrac{\pi}{5} - \cot \tfrac{2\pi}{5})e^{\frac{\pi i}{5}}\psi + O(\psi^2) \right\}, \tag{86}$$

as a convergent power series for $|\psi| < 1$. We may now map the entire $\mathbb{P}^1$ of $\mathscr{M}_A(X)$ into the $(B + iJ)_1$-plane. The result is shown in figure 4.

The first striking feature of figure 4 is that not all values of J_1 are allowed. The sphere is mapped to the striped region and so J_1 acquires a minimum value of $\frac{1}{2} \cot \frac{\pi}{5}$. This fits in with the idea of a minimum allowed size as we saw for the circle. Any conformal field theory in this class may be associated with a Calabi-Yau manifold of a size scale no less than the order of the Planck length. The shaded area in figure 4 represents the area in which the instanton sum for the Calabi-Yau nonlinear σ-model converges. Depending on one's taste, one might wish to declare this to set the minimum length and then say that the bottom region must be describe in terms of Landau-Ginzburg orbifolds.

A very important point to note is the following. The striped region in figure 4 may at first sight look similar to a fundamental $Sl(2,\mathbb{Z})$ region in the upper half plane as we had for the torus example in section 3. Sadly the region in figure 4 *cannot* be obtained this way. One may map this region by a transcendental function to another region which may be written in the form of a fundamental region [47] but as this map is not one-to-one it is not
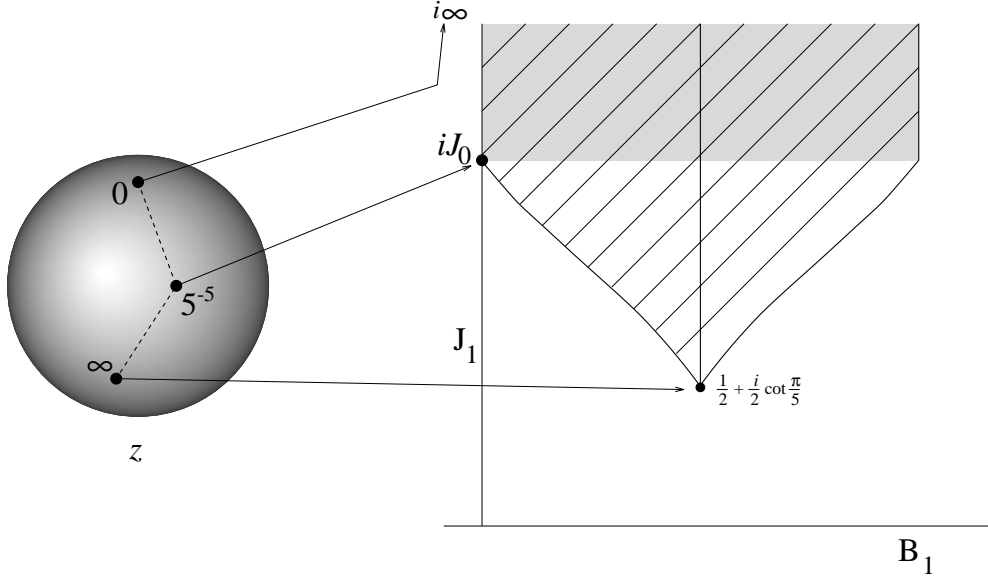
Figure 4: Mapping $\mathscr{M}_A(X)$ to the $(B + iJ)_1$-plane.

clear what purpose this can serve for a model of the moduli space. The moduli spaces for the torus and for K3 surfaces appeared naturally in the form of a Teichmüller space divided by some modular group. It appears that for more generic $N{=}2$ theories such a description is no longer valid. This should not dishearten the reader however — we were still able to find $\mathscr{M}_A(X)$ without such a description.

6   Phases

In the last section we looked at the simplest case of a moduli space for a nontrivial case $d = 3$. It was found that the space naturally divided into two phases. In this section we will discuss the more general picture.

6.1   Another Example

To facilitate the discussion it will prove useful to run through another example. This example was first analyzed in [57] (see also [58]). The idea is very similar to that of the quintic hypersurface except this time we begin with a *weighted* projective space $\mathbb{P}^4_{\{2,2,2,1,1\}}$. This is the space with homogeneous coordinates $[x_0, \ldots, x_4]$, omitting $[0, 0, 0, 0, 0]$, where we identify

$$[x_0, x_1, x_2, x_3, x_4] \cong [\lambda^2 x_0, \lambda^2 x_1, \lambda^2 x_2, \lambda x_3, \lambda x_4], \tag{87}$$

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where $\lambda \in \mathbb{C}^*$. The desired Calabi-Yau hypersurface is given by

$$f = x_0^4 + x_1^4 + x_2^4 + x_3^8 + x_4^8 = 0. \tag{88}$$

This hypersurface is not smooth. There is a $\mathbb{Z}_2$ singularity along the surface $[x_0, x_1, x_2, 0, 0]$ in $\mathbb{P}^4_{\{2,2,2,1,1\}}$ which intersects our hypersurface along a curve. At each point on the curve, this singularity is essentially the same as the the one studied in section 4. Thus we may blow each point of the curve up to get $\mathbb{P}^1$ to smooth the space. The resulting exceptional divisor, E, will be a surface (basically the old curve$\times \mathbb{P}^1$). This smooth Calabi-Yau manifold, containing the surface E, will be our smooth model for X. The hyperplane in $\mathbb{P}^4_{\{2,2,2,1,1\}}$ produces another divisor in X which we call F. The divisors E and F are associated to two linearly independent elements of $H^2(X)$. In fact $h^{1,1}(X) = 2$ and these elements form a basis. Therefore $\mathscr{M}_A(X)$ is now two-dimensional.

For Witten's linear σ-model we now want $s = 2$ and so we need two $U(1)$ charges and on complexification have two $\mathbb{C}^*$ actions to consider on the target space. We assign the following charges:

ϕ_i	$Q_i^{(1)}$	$Q_i^{(2)}$
x_0	0	1
x_1	0	1
x_2	0	1
x_3	1	0
x_4	1	0
s	-2	1
p	0	-4

$$\tag{89}$$

and have the following superpotential

$$W = p(x_0^4 + x_1^4 + x_2^4 + s^4 x_3^8 + s^4 x_4^8 + \ldots), \tag{90}$$

where ... represents other term that may be added (with arbitrary coefficients) respecting the quasi-homogeneity of the equation.

The D-terms are now fixed by the equations of motion as

$$D_1 = |x_3|^2 + |x_4|^2 - 2|s|^2 - r_1,$$
$$D_2 = |x_0|^2 + |x_1|^2 + |x_2|^2 + |s|^2 - 4|p|^2 - r_2. \tag{91}$$

Recall that for the example in section 5.3 we found that fields which were sections of $\mathcal{O}(n)$ vanished when $n \leq 0$ even though one might initially expect this only to be the case when $n < 0$. The $n = 0$ case should be checked by the transversality condition. We assume this to be the case below except where noted.

We obtain four phases:

1. $r_1 < 0$ and $r_1 + 2r_2 < 0$:
   In this case $n_1 \leq \frac{1}{2}n_2$ and $n_2 \leq 0$ by the condition $\sum_l n_l r_l \geq 0$. The fields x_0, x_1, x_2 are sections of $\mathcal{O}(n_2)$ and the fields x_3, x_4 are sections of $\mathcal{O}(n_1)$. Therefore they all vanish. Taking the D-terms to vanish to solve $U = 0$ forces a non-zero value for s and p which may be fixed by the $(\mathbb{C}^*)^2$ gauge group. Thus the target space is a point. There is a residual $\mathbb{Z}_8$ symmetry left over so that our theory is actually a Landau-Ginzburg theory with target space $\mathbb{C}^5/\mathbb{Z}_8$.

2. $r_1 < 0$ and $r_1 + 2r_2 > 0$:
   In this case $n_2 \geq 0$ which forces $p = 0$. Solving for $U = 0$, the D-terms force s to be non-zero and that $x_0, \dots, x_4$ cannot simultaneously vanish. We may use one $\mathbb{C}^*$ symmetry to fix s. The other one may be used to turn $x_0, \dots, x_4$ into the homogeneous coordinates of $\mathbb{P}^4_{\{2,2,2,1,1\}}$. The target space is thus the hypersurface $f = 0$ in $\mathbb{P}^4_{\{2,2,2,1,1\}}$. We call this the "orbifold" phase since this hypersurface has $\mathbb{Z}_2$ quotient singularities.

3. $r_1 > 0$ and $r_2 < 0$:
   Now $n_1 \geq 0$ forcing $s = 0$ and $n_2 \leq 0$ forcing $x_0 = x_1 = x_2 = 0$. Solving for $U = 0$ forces p to be nonzero and x_3 and x_4 to not simultaneously vanish. One $\mathbb{C}^*$ may be used to fix p and the other to turn x_3, x_4 into homogeneous coordinates on $\mathbb{P}^1$. The target space is therefore $\mathbb{P}^1$. This is not quite the full story however, the fluctuations of the fields x_0, x_1, x_2, s are massless and governed by a quartic superpotential. Fixing p leaves a residual $\mathbb{Z}_4$ symmetry meaning that that fields x_0, x_1, x_2, s live in the space $\mathbb{C}^4/\mathbb{Z}_4$. This phase is a hybrid-like phase consisting of a Landau-Ginzburg (orbifold) bundle over $\mathbb{P}^1$. We call this the "$\mathbb{P}^1$ phase".

4. $r_1 > 0$ and $r_2 > 0$:
   This case is a little more complicated and we have to be more careful concerning our assertion that sections of $\mathcal{O}(m)$ are forced to vanish when $m \leq 0$. We have $n_2 \geq 0$ which will force $p = 0$ when $n_2 > 0$ and we will find that $p = 0$ by transversality when $n_2 = 0$. When $n_1 > 0$ we will have $s = 0$ but we need to exercise more care when $n_1 = 0$. Solving for $U = 0$ we find that x_3, x_4 cannot simultaneously vanish and that x_0, x_1, x_2, s cannot simultaneously vanish. First assume that $s \neq 0$. We may then use one of the $\mathbb{C}^*$-actions to fix s. Let the other $\mathbb{C}^*$ make $x_0, \dots, x_4$ into homogeneous coordinates. Our target space is now very similar to that of the hypersurface in $\mathbb{P}^4_{\{2,2,2,1,1\}}$ except that we are missing the points where $x_3 = x_4 = 0$ and $x_0^4 + x_1^4 + x_2^4 = 0$. Now let $s = 0$. Now we have $x_0^4 + x_1^4 + x_2^4 = 0$ and x_3 and x_4 may be any value except both zero. Use one of the $\mathbb{C}^*$-actions to turn x_3, x_4 into the homogeneous coordinates of $\mathbb{P}^1$ and the other $\mathbb{C}^*$-action turn x_0, x_1, x_2 into homogeneous coordinates. What we have is a description of the smooth, blown-up Calabi-Yau manifold described above — the singular point set has been replaced by an exceptional divisor consisting of a curve times $\mathbb{P}^1$. This is the "Calabi-Yau" phase.
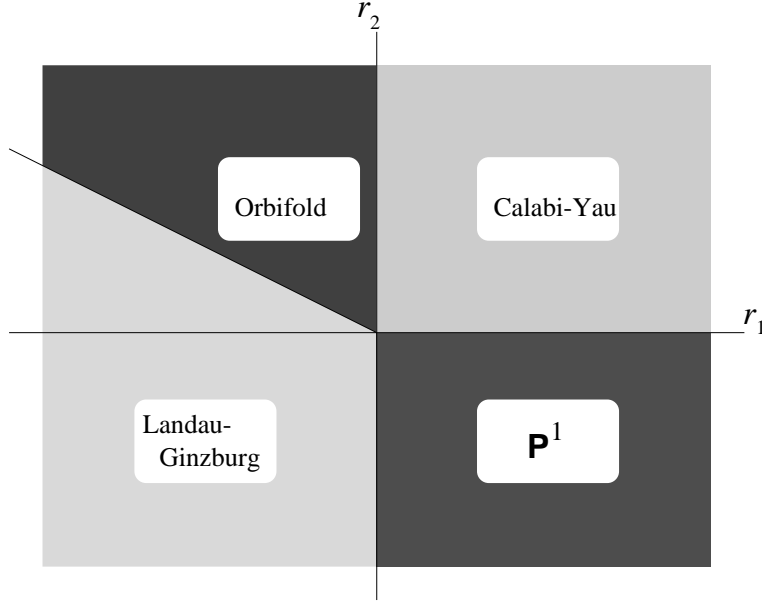
Figure 5: Phase diagram for $h^{1,1} = 2$ example.

This produces a "phase diagram" which we show in figure 5.

The instantons which appear in each sector can now appear in various forms. The cases are given as follows:

1. For the Landau-Ginzburg orbifold we have only twist-fields.

2. For the orbifold phase we have two kinds. Firstly we have rational-curve type instantons (together with their small-scale supplements). Secondly we have twist-field instantons confined to the region around the $\mathbb{Z}_2$-quotient singularity.

3. For the $\mathbb{P}^1$ phase we have rational curve type instantons — the rational curve being $\mathbb{P}^1$ itself! We also have twist-field instantons in the Landau-Ginzburg fibre since this has a $\mathbb{Z}_4$ quotient singularity.

4. For the Calabi-Yau phase we have rational curve instantons.

Now let us consider the mirror, Y, of this example. This is obtained by dividing X by the group $(\mathbb{Z}_4)^3$ consisting of elements

$$g \colon (x_0, \ldots, x_4) \mapsto (e^{2\pi i s_0} x_0, \ldots, e^{2\pi i s_4} x_4), \tag{92}$$

where $4s_0, 4s_1, 4s_2, 8s_3, 8s_4$ and $s_0 + \ldots + s_4$ are all integers. The general form of the defining equation for Y is then

$$f = x_0^4 + x_1^4 + x_2^4 + x_3^8 + x_4^8 - 8\psi x_0 x_0 x_0 x_0 x_0 - 2\phi x_3^4 x_4^4, \tag{93}$$
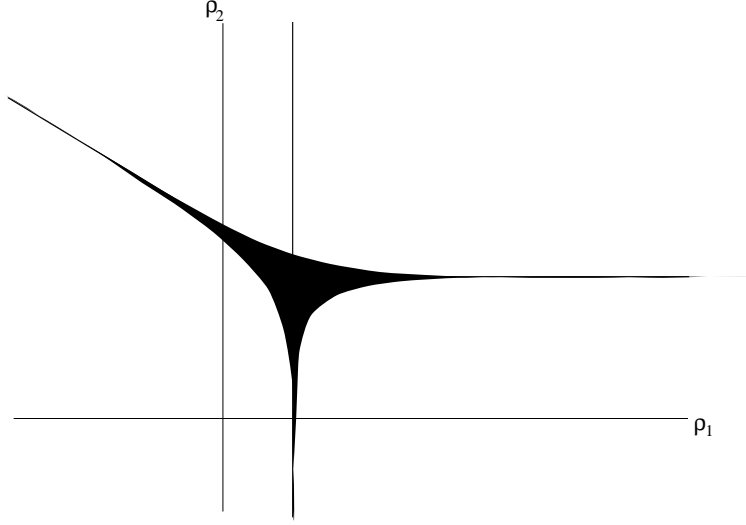
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Figure 6: Singular locus for $h^{1,1} = 2$ example.

where ϕ and ψ are complex parameters that vary the complex structure of Y. This is in agreement with our expectation that $h^{2,1}(Y) = 2$. To map the space spanned by ϕ and ψ to the moduli space of linear σ-models for X we need to find special points in these spaces. We first consider the values of ψ and ϕ for which the hypersurface (93) becomes singular. A little algebra shows that $\partial f / \partial x_i = 0$ for all x_i admits nontrivial solution when $\phi^2 = 1$ or $(\phi + 8\psi^4)^2 = 1$. With some foresight let us introduce the variables

$$
\begin{aligned}
\rho_1 &= \tfrac{1}{2\pi} \log |4\phi^2|, \\
\rho_2 &= \tfrac{1}{2\pi} \log \left| \frac{2^{11}\psi^4}{\phi} \right|.
\end{aligned}
\tag{94}
$$

We may now map the singular points in our moduli space into the ρ_1, ρ_2 plane. That is we find the values for ρ_1 and ρ_2 for which there can be a ϕ and ψ which give a singular Y. The result is shown in figure 6

Now imagine that we "zoom out" infinitely far from figure 6. It should be easy to see that the shaded region will become precisely the phase boundaries in figure 5. In fact, the variables in (94) were chosen precisely so that we may make the identification $r_1 = \rho_1$ and $r_2 = \rho_2$. The fact that the asymptotes of the region in figure 6 are parallel to, rather than along, lines passing through the origin comes from the one-loop renormalization effects of r_l — just as we had a $5 \log 5$ shift in section 5.3. Thus assuming a simple map between the B-model moduli space coordinates and the linear σ-model coordinates as we had in (79), we

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obtain

$$z_1 = \frac{1}{4\phi^2},$$

$$z_2 = \frac{\phi}{2^{11}\psi^4}.$$

(95)

The reader is referred to [45] for the complete proof that this is indeed the correct map.

Again we may calculate the Picard-Fuch's equation for this two-parameter example [57]. Again (69) uniquely determines the relationships

$$q_1 = z_1 + 2z_1^2 + 48z_1z_2 + 5z_1^3 + 7560z_1z_2^2 + \ldots$$

$$q_2 = z_2 - z_1z_2 + 104z_2^2 - z_1^2z_2 - 56z_1z_2^2 + 15188z_2^3 + \ldots$$

(96)

This is a convergent power series in the Calabi-Yau phase.

As before each phase leads to some notion of a series for correlation functions which will be convergent in some region. In the example studied in section 5.3 we found that any generic point in the moduli space lay in the region of convergence of one of the power series. This is no longer the case in this two parameter example. The shaded region in figure 6 marks where none of the power series converge. In this sense, the phase picture does not cover the entire moduli space although it will cover most of a circle with center $r_1 = r_2 = 0$ in the limit of infinite radius.

As we did in section 5.3 we may analytically continue the q_l's into the other three phases. This calculation was done in [59]. Doing so one obtains the following bound for the entire moduli space

$$\boxed{J_1 \geq 0.}$$

(97)

This is a very important result. It means that the algebraic curves in X whose areas are measured by J_1 (i.e., the rational curves in the exceptional divisor) may shrink down to zero size. **We have broken the constraint of a minimum length.**

The reader may argue that we analytically continued q_l beyond the Calabi-Yau phase and so we may artificially have been able to reach zero size. Actually in this case, J_1 can take on arbitrarily small values while one remains in the Calabi-Yau phase [59]. Thus one really is forced to reject a notion of minimum length in this example.

It is worth noting that general arguments have been made which appear to prove a universal notion of minimum length (such as [60]). These arguments appear to rely on the assumption that space-time is locally flat, i.e., one believes the differential geometry view. Unless we can concoct some alternative view of small distances this will appear to show that differential geometry is seriously misleading at small scales!

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6.2 More general cases

The examples studied so far show many of the aspects of the global structure of the moduli space — $\mathscr{M}_B(X)$ is rather dull and can be explained classically while $\mathscr{M}_A(X)$ forces a "phase" description upon us and forces us to rethink our notions of geometry at small scales. Let try to understand the more general picture of $\mathscr{M}_A(X)$.

One of the phase transitions for the example studied in section 6.1 was between an orbifold and a Calabi-Yau manifold. It was precisely the blow-up of section 4. All of the phase transitions mentioned thus far are actually of this type.

Let us return to the example of the quintic threefold of section 5.3. The two phases here were a smooth Calabi-Yau manifold and an Landau-Ginzburg theory in $\mathbb{C}^5/\mathbb{Z}_5$. The natural question to ask is if we can blow-up the quotient singularity $\mathbb{C}^5/\mathbb{Z}_5$. The answer is yes, and the procedure is very similar to that of blowing up the $\mathbb{C}^2/\mathbb{Z}_2$ singularity. In the latter case we used the line bundle over $\mathbb{P}^1$ with first Chern class -2. For $\mathbb{C}^5/\mathbb{Z}_5$ we use the line bundle over $\mathbb{P}^4$ with first Chern class -5. That is, the exceptional divisor is $\mathbb{P}^4$ — but $\mathbb{P}^4$ is precisely the ambient space for the Calabi-Yau phase. This leads us to the following general picture. Define X as the critical point set of some function W on some (non-compact) space, V. By blowing up and down quotient singularities in V we induce the phase transitions in X.

The singularity $\mathbb{C}^5/\mathbb{Z}_8$ is blown up using an exceptional divisor with two irreducible components. Having each of these components blown up or down leads to the 4 phases. For example, the first component is $\mathbb{P}^1 \times \mathbb{C}^3$. Blowing up the Landau-Ginzburg phase using this component gives the $\mathbb{P}^1$ phase. This still has $\mathbb{Z}_4$ singularities all along this $\mathbb{P}^1$. The second component is $\mathbb{P}^4$. This component resolves the $\mathbb{Z}_4$ singularities in each fibre. The restriction to the critical point set of W in each fibre forms a quartic constraint. A quartic in constraint in $\mathbb{P}^4$ is a K3 surface. Thus this latter phase is a fibre bundle over $\mathbb{P}^1$ with generically a K3 fibre. This is one description of the smooth Calabi-Yau manifold [57]. We may also reach the smooth phase by blowing up along $\mathbb{P}^4$ first to obtain the orbifold phase and then along $\mathbb{P}_1 \times \mathbb{C}^3$ to resolve the orbifold.

Have we constructed the general picture by considering blow-ups of quotient singularities? The answer to this question is no. We should then ask the question as to whether there is any transformation that might give the general picture. The answer that appears to be the case is that we consider "birational" transformations. Birational transformations occur very naturally in algebraic geometry and thus we shouldn't be too surprised that they will be natural objects in N=2 theories. Two algebraic varieties X_1 and X_2 are birationally equivalent if one can find open subsets $U_1 \subset X_1$ and $U_2 \subset X_2$ such that the set of functions in U_1 is isomorphic to the set of functions in U_2 (see [61] for a more careful definition). An example of birationally equivalent pairs are given by X_1 being a blow-up of X_2.

Another example of birational equivalence can be provided still from quotient singularities. This stems from the fact that the process of blowing-up a quotient singularity need not be a unique process. Suppose we have a singular space X_0 which may be smoothed by

blowing-up into two topologically distinct smooth spaces X_1 and X_2. These two smooth spaces are then birationally equivalent.

In [62, 56] an example with $h^{1,1}(X) = 5$ was studied. This requires toric geometry which takes us beyond the scope of these lectures so we will not provide details here. There are 100 phases in all for this model of which 5 consist of smooth Calabi-Yau manifolds. These 5 manifolds are all the possible resolutions of an orbifold which provides one of the other phases.

One may also picture a direct transition between these manifolds in the form of a "flop" as we now explain. We can raise our example of a $\mathbb{C}^2/\mathbb{Z}_2$ quotient singularity of section 4 to one higher dimension by considering the space $xy - wz = 0$ in $\mathbb{C}^4$ with coordinates (x, y, z, w). Transversality tells us that this hypersurface has an isolated singularity at $(0, 0, 0, 0)$. It cannot be written as a quotient singularity. Now consider the space

$$\mathcal{O}(-1, -1) = \{[a, b], (x, y, z, w) \in \mathbb{P}^1 \times \mathbb{C}^4; \, az = bx, ay = bw\}. \tag{98}$$

This smooths the singularity, in a manner similar to blow-ups discussed earlier, replacing the singular point by $\mathbb{P}^1$. The space

$$\mathcal{O}(-1, -1) = \{[a, b], (x, y, z, w) \in \mathbb{P}^1 \times \mathbb{C}^4; \, aw = bx, ay = bz\} \tag{99}$$

does pretty much the same thing. The only difference between (98) and (99) is the way in which the $\mathbb{P}^1$ is inserted. The two smooth spaces produced are said to differ by a flop. This local picture may be fitted into more complicated global geometries.

These five smooth models for X occupy adjacent phases (since they are related by flops) in the phase picture and so we may consider a path of conformal field theories passing from one to the other. So long as this path is generic, it will not hit the singularity lying in the complex codimension one "phase boundary" and so this transition is perfectly smooth from the conformal field theory point of view. This shows how string theory can give rise to a "smooth topology change" in the target space so long as the two topological spaces are birationally equivalent as algebraic varieties.

If the reader was not convinced thus far that algebraic geometry was superior to differential geometry for our purposes then this last point must surely convince them. The natural equivalence class in differential geometry is that of diffeomorphic equivalence which is a class stronger than topological equivalence. String theory happily combines different classes smoothly into the same moduli space — string theory is oblivious to such distinctions! However, every phase in a given moduli space belongs to single birational equivalence class.

There are many more possibilities of birational transformations than those provided by resolving quotient singularities and flops. One of the more exotic ones is the "exoflop" of [56]. In the case of the flop, one $\mathbb{P}^1$ is transformed into another one within a Calabi-Yau manifold. In the the case of an exoflop, a rational curve is taken from within the Calabi-Yau manifold to one glued onto the outside.

The complete picture of possible phases is far from complete. The most general construction for models for which one understands the linear σ-model is that provided by [38]. These are explored to some extent [63] although the full range of possibilities have yet to be classified. One interesting point discussed in [63] is that the phase diagrams need not contain any smooth Calabi-Yau phase. This really is quite natural given that there is nothing particularly special about the smooth Calabi-Yau phase — it has no more reason to exist than any of the other phases. The reason we tend to like Calabi-Yau manifolds rather than the other phases is because we tend to be happier with differential geometry which only works properly in this phase.

Having said that we should rid our minds of any bias towards the Calabi-Yau manifold phase we should note some rather interesting curiosities which occur when we try to analyze the whole moduli space in terms of these smooth phases. Let us consider the five parameter example of [56] which has 100 phases of which 5 are Calabi-Yau manifolds. When we do the analytic continuation analysis to map out the entire moduli space in terms of one phase, which Calabi-Yau phase should we begin with? The surprising answer is that is doesn't actually matter as we now explain.

Suppose we begin with one of the topologies for X which we denote X_1. Thus we have some phase in the moduli space containing the large radius limit of X_1 where we put $z_1 = \ldots = z_5 = q_1 = \ldots = q_5 = 0$. Now let X_2 be another topology which is obtained by flopping X_1. This flop will correspond to a rational curve $C_1 \subset X_1$ and another rational curve $C_2 \subset X_2$ which arise in the form of equations (98) and (99). We may chose q_5 such that

$$\text{Area}(C_1) = J_5. \tag{100}$$

Now we consider the flop process. To make things simple we should make every rational curve in X_1 have infinite area except for C_1. This may be achieved by setting $q_1 = \ldots = q_4 = 0$. Now we analyze the Picard-Fuch's equation to obtain q_5 in terms of z_5. The result is [64]

$$q_5 = z_5. \tag{101}$$

Analysis of the phase picture shows that if we introduce coordinates $z'_1, \ldots, z'_5$ for a patch of coordinates with origin at the large radius limit of X_2 then $z'_5 = (z_5)^{-1}$. We also have $q'_5 = z'_5$. This means that the analytic continuation between these two phases simply asserts that the area of C_1 is minus the area of C_2. Thus we naturally identify a topology with a certain conformal field theory by demanding that all areas be positive.

It appears to be the case [64] that the following now applies in the case that there is at least one Calabi-Yau phase. Given any conformal field theory we may associate some Calabi-Yau target space topology (possibly by analytic continuation) in which all areas are non-negative. That is to say when we go from a phase picture in terms of r_l's which covers the whole space $\mathbb{R}^s$ and then remap this space into the space of J_l's then only points within (or on the boundary of) cones corresponding to smooth Calabi-Yau phases are covered. This
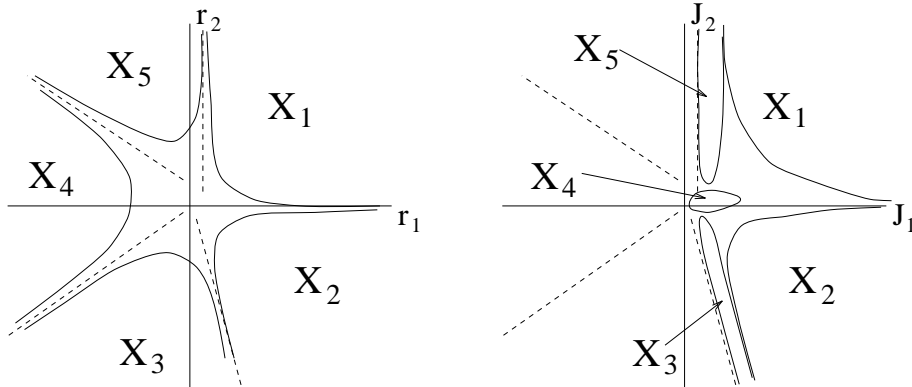
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Figure 7: Moving between r_l and J_l.

is shown schematically in figure 7 for a hypothetical example with $s = 2$ and five phases $X_1, \ldots, X_2$ for which only X_1 and X_2 are smooth Calabi-Yau manifolds.

7   Conclusions

We hope the reader is convinced that algebraic geometry together with mirror symmetry is a very useful tool for analyzing the moduli space of $N=2$ theories. Although sticking to metric-based ideas seems to work without any problems for some models, it appears to face severe short-comings in the generic case. In particular, when there are instantons one tends to have phases since one will have instanton sums which are only convergent in a certain region of the moduli space. When this happens the other phases appear to have an equal right to be taken as a model for the target space and then one necessarily needs to discuss singular spaces.

Where metric-based ideas may work, in the case $d = 3$, appears to be restricted to the few cases where $h^{2,0} > 0$ (which implies that $h^{1,0} > 0$ which in turn implies that the target space has a continuous isometry). Although we didn't discuss these examples in detail, the interested reader may try to apply the methods of section 3 and 4 for these cases.

The methods used in section 5 have certain shortcomings which may not have been completely apparent. Firstly there may be some subtleties introduced into the phase picture if there are some symmetries of the defining equation that are not of the most naïve kind. We refer the reader to [65] for such an example. Another, more severe, problem is that not all of the dimensions of the moduli space may be parametrized by the linear σ-model. That is, we may only be able to write down models based on a gauge group $U(1)^s$ where $s < h^{1,1}(X)$. The mirror to this statement is the fact that not all deformations of a complete intersection can, in general, be written as deformations of the defining equations [18]. We do not have any techniques at our disposal to address the complete moduli spaces of such

objects at this point in time.

We should also repeat that some $N=2$ conformal field theories cannot be put in the form of a gauged $U(1)$ linear σ-model at all. This is equivalent to the statement in geometry that not all Calabi-Yau manifolds can be written as complete intersections in toric varieties. One may try to extend methods to nonabelian gauge groups. Although such models were discussed in [2], since the mirror map construction for such models is not yet understood, we may not apply most of the methods discussed in this paper. Also there is no reason to believe that these nonabelian groups will exhaust all the $N=2$ theories.

Although the moduli space of $N=2$ theories is far better understood now than people imagined a few years ago it appears that there is still much left to discover.

Acknowledgements

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